Design of Two-Channel Equiripple FIR Linear-Phase Quadrature Mirror Filters Using the Vector Space Projection Method

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***Abstract*— A new technique for designing finite impulse response (FIR) quadrature mirror filters (QMF's) is proposed. The approach is based on the vector space projection method (VSPM). Convex constraint sets and their projections that capture the properties of the desired QMF's are given. The proposed approach produces equiripple filters. This method is compared with the classical Johnston's method and is shown to have certain advantages.**

I. Introduction

QUADRATURE mirror filters (QMF's) [1] are used in subband coders for speech processing [2], transmultiplexers for telecommunication [3], and image compression [4]. A common requirement in most applications is that the reconstructed output signal should be "as close" as possible to the input signal. When the output is a delayed, replica of the input then the QMF's system is called a *perfect reconstruction* (PR) QMF system. In order to design a PR QMF, one needs to relax one of the QMF properties, such as the phase linearity or the power complementary property. On the other hand, it may be desirable that the analysis and synthesis portions of the QMF's have linear-phase for certain applications that deal (for example) with low bit rate coding.

A classical method for designing near perfect reconstruction QMF's was first proposed by Johnston [5]. It consist of selecting the filter coefficients such that $|H(\omega)|^2+|H(\pi-\omega)|^2$ is made as close to unity as possible (the so-called power complementary property) while simultaneously minimizing (or constraining) the stopband energy of the transfer function $H(\omega)$. This approach leads to the minimization of the integral squared error

$$J=\mu\int_{\omega_s}^{\pi}|H(\omega)|^2\,d\omega + (1-\mu)\int_0^{\pi}[|H(\omega)|^2+|H(\pi-\omega)|^2-1]^2\,d\omega$$

where μ is a weighting factor in the range of $0<\mu<1$ and ω_s is the stopband frequency. In performing the optimization, the filter impulse response is constrained to be symmetric (linear phase). In this letter, we propose a new approach to designing QMF's based on the vector space projection method (VSPM). We will provide examples and compare this method with Johnston's method.

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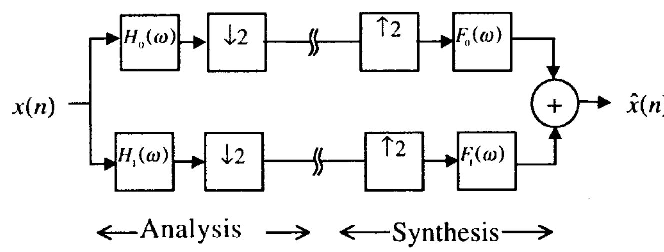

Fig. 1. Two-channel QMF bank.

VSPM deals with the problem of finding an object (for example, a signal, function, image, etc.) in a proper vector space that satisfies multiple constraints. When all the constraint sets are convex and have a *nonempty intersection,* there exists a powerful theory in finding the object that satisfies all the constraints. This subset of VSPM is called *projection onto convex sets* (POCS). The reader is referred to [6] and [7] for an introduction to this method.

II. Two-Channel QMF Properties

A two-channel QMF bank is shown in Fig. 1, where $H_0(\omega)$, $H_1(\omega)$ are the lowpass and highpass filters, respectively, of the analysis bank filters, and $F_0(\omega)$, $F_1(\omega)$ are the corresponding synthesis filters. The conditions imposed on the analysis and synthesis filters for a PR system are well known; see, for example, [1]. All filters are assumed linear phase and are of length N, where N is even. For the analysis sections, the impulse responses are

$$h_0(n)=h_0(N-n-1)\quad \text{for } 0\le n\le N/2-1 \tag{1}$$
$$h_1(n)=(-1)^n h_0(n) \tag{2}$$
$$h_1(n)=-h_1(N-n-1)\quad \text{for } 0\le n\le N/2-1 \tag{3}$$

$$|H_0(\omega)|^2+|H_1(\omega)|^2=1\quad \text{for } 0\le\omega\le\pi. \tag{4}$$

And for the synthesis section, the responses are

$$f_0(n)=2h_0(n),\quad f_1(n)=-2(-1)^n h_0(n). \tag{5}$$

In addition to the above conditions, it is desirable that the filters have the passbands close to unity and good attenuation in the stopbands. As mentioned previously one needs to relax the power complementary property to design a QMF with linear phase. In what follows, we propose a design algorithm based on VSPM.

III. Design of Two-Channel Linear Phase QMF Using VSPM

The first step in implementing the VSPM algorithm is to define the appropriate sets that capture the QMF's properties. These sets are parametrized by the constraints needed to specify the characteristics of the QMF's. In parallel with (1) to (4) and taking into consideration the stopbands attenuation we define the following appropriate sets. In the frequency domain, the sets are (6)–(8), shown at the bottom of the page, where ω_1 is the passband and stopband frequency of $H_0(\omega)$ and $H_1(\omega)$, respectively, and $\omega_2 = \pi - \omega_1$ is the passband and stopband frequency of $H_1(\omega)$ and $H_0(\omega)$, respectively. Since the lengths of $\mathbf{h}_0$ and $\mathbf{h}_1$ are known, $\varphi_0(\omega)$ and $\varphi_1(\omega)$ the linear phases associated with $H_0(\omega)$ and $H_1(\omega)$ are also known. In the time domain, the set is (9), shown at the bottom of the page.

In words, C_1 is the set of all two-tuple finite-length sequences that imply a Fourier transform that satisfies (4) (power complementarity) with an error tolerance of 2σ. Set C_2 is the set of all two-tuple, finite-length sequences, whose lowpass response $\mathbf{h}_0 \leftrightarrow H_0(\omega)$ has a stopband attenuation magnitude bounded by δ. In $C_2, \mathbf{h}_1$ is otherwise unrestricted. Set C_3 is the set of all two-tuple, finite-length sequences, whose highpass response $\mathbf{h}_1 \leftrightarrow H_1(\omega)$ has a stopband attenuation magnitude bounded by δ. In $C_3, \mathbf{h}_0$ is otherwise unrestricted. Set C_4 is the set of symmetrical sequences $\mathbf{h}_0$, and $\mathbf{h}_1$ that satisfies (1), (2), and (3) (the QMF property and linear phase). The convexity of C_1, C_2, C_3, and C_4 can be easily established using similar arguments as for the sets defined in [6, pp. 225–228].

The next step is to find the projections onto these sets. In the interest of brevity, we give projections without proof.

Projection onto C_1: The projection of an arbitrary two-tuple $(\mathbf{g}_0, \mathbf{g}_1) \in \mathbf{R}^N \times \mathbf{R}^N \leftrightarrow (G_0(\omega), G_1(\omega))$ onto C_1 where $G_i(\omega) = |G_i| e^{j\varphi_i(\omega)}$, $i = 0, 1$ can be computed using the Lagrange multiplier method. The results are as follows.

If $|G_0(\omega)|^2 + |G_1(\omega)|^2 > 1 + \sigma$, the projection P_1 onto C_1 will be

$$(\mathbf{h}_0^*, \mathbf{h}_1^*) = P_1(\mathbf{g}_0, \mathbf{g}_1) \leftrightarrow (H_0^*(\omega), H_1^*(\omega)) \tag{10}$$

where

$$(H_0^*(\omega), H_1^*(\omega)) = \left(\frac{|G_0(\omega)|}{\sqrt{\frac{|G_0(\omega)|^2 + |G_1(\omega)|^2}{1+\sigma}}} e^{j\varphi_0(\omega)} \quad \frac{|G_1(\omega)|}{\sqrt{\frac{|G_0(\omega)|^2 + |G_1(\omega)|^2}{1+\sigma}}} e^{j\varphi_1(\omega)} \right). \tag{11}$$

If $|G_0(\omega)|^2 + |G_1(\omega)|^2 < 1 - \sigma$, the projection P_1 onto C_1 will be

$$(\mathbf{h}_0^*, \mathbf{h}_1^*) = P_1(\mathbf{g}_0, \mathbf{g}_1) \leftrightarrow (H_0^*(\omega), H_1^*(\omega)) \tag{12}$$

where

$$(H_0^*(\omega), H_1^*(\omega)) = \left(\frac{|G_0(\omega)|}{\sqrt{\frac{|G_0(\omega)|^2 + |G_1(\omega)|^2}{1-\sigma}}} e^{j\varphi_0(\omega)} \quad \frac{|G_1(\omega)|}{\sqrt{\frac{|G_0(\omega)|^2 + |G_1(\omega)|^2}{1-\sigma}}} e^{j\varphi_1(\omega)} \right). \tag{13}$$

Finally, if $1 - \sigma \le |G_0(\omega)|^2 + |G_1(\omega)|^2 \le 1 + \sigma$, the projection P_1 onto C_1 will be

$$(\mathbf{h}_0^*, \mathbf{h}_1^*) = P_1(\mathbf{g}_0, \mathbf{g}_1) \leftrightarrow (H_0^*(\omega), H_1^*(\omega)) \tag{14}$$

where

$$(H_0^*(\omega), H_1^*(\omega)) = (G_0(\omega), G_1(\omega)). \tag{15}$$

These equations implicitly define the projector P_1.

Projection onto C_2: The projection of an arbitrary $(\mathbf{g}_0, \mathbf{g}_1) \leftrightarrow (G_0(\omega), G_1(\omega))$ is obtained using the Lagrange multiplier method. The result is $(\mathbf{h}_0^*, \mathbf{h}_1^*) = P_2(\mathbf{g}_0, \mathbf{g}_1)$ where

$$\mathbf{h}_0^* \leftrightarrow \begin{cases} \delta[G_0(\omega)/|G_0(\omega)|], & \text{if } |G_0(\omega)| > \delta, \text{ for } \omega_2 \le \omega \le \pi, \\ G_0(\omega), & \text{if } |G_0(\omega)| \le \delta, \text{ for } \omega_2 \le \omega \le \pi, \\ G_0(\omega), & \text{elsewhere.} \end{cases}$$
$$\mathbf{h}_1^* \leftrightarrow G_1(\omega). \tag{16}$$

$$C_1 \triangleq \left\{ \begin{matrix} (\mathbf{h}_0, \mathbf{h}_1) \in \mathbf{R}^N \times \mathbf{R}^N : 1 - \sigma \le \left|H_0^2(\omega)\right| + \left|H_1^2(\omega)\right| \le 1 + \sigma \\ \text{and } \varphi_0(\omega) = \varphi_1(\omega) = \omega(N-1)/2 \quad \text{for } 0 \le \omega \le \pi \end{matrix} \right\} \tag{6}$$

$$C_2 \triangleq \{(\mathbf{h}_0, \mathbf{h}_1) \in \mathbf{R}^N \times \mathbf{R}^N : |H_0(\omega)| \le \delta \quad \text{for } \omega_2 \le \omega \le \pi\} \tag{7}$$

$$C_3 \triangleq \{(\mathbf{h}_0, \mathbf{h}_1) \in \mathbf{R}^N \times \mathbf{R}^N : |H_1(\omega)| \le \delta \quad \text{for } 0 \le \omega \le \omega_1\} \tag{8}$$

$$C_4 \triangleq \left\{ \begin{matrix} (\mathbf{h}_0, \mathbf{h}_1) \in \mathbf{R}^N \times \mathbf{R}^N : h_0(n) = h_0(N-n-1), h_1(n) = h_1(N-n-1) \\ \text{and } h_1(n) = (-1)^n h_0(n) \quad \text{for } n = 0, 1, \ldots, N/2 - 1 \end{matrix} \right\} \tag{9}$$

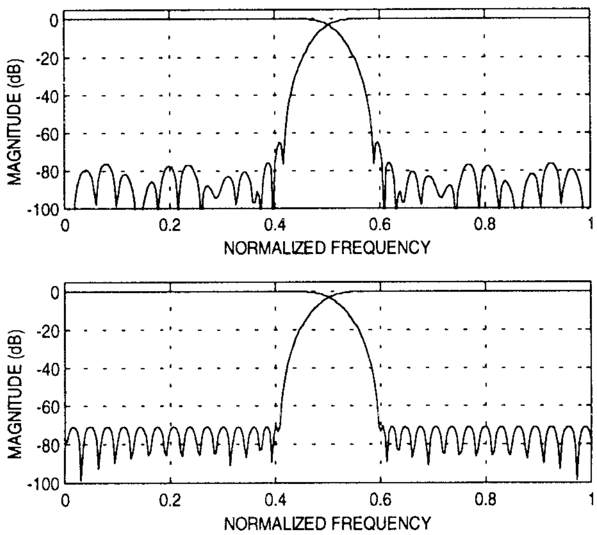

Fig. 2. Frequency response for the Johnston (top) and VSPM (bottom) analysis filters.

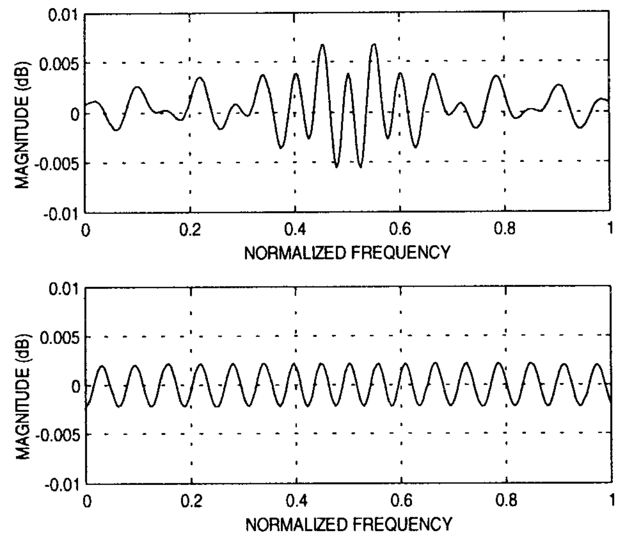

Fig. 3. Plot of $|H_0(\omega)|^2 + |H_1(\omega)|^2$ for the Johnston (top) and VSPM (bottom) pair of filters.

Projection onto C_3: The projection of an arbitrary $(\mathbf{g}_0, \mathbf{g}_1) \leftrightarrow (G_0(\omega), G_1(\omega))$ is obtained using the Lagrange multiplier method. The result is $(\mathbf{h}_0^*, \mathbf{h}_1^*) = P_3(\mathbf{g}_0, \mathbf{g}_1)$ where

$$\mathbf{h}_0^* \leftrightarrow G_0(\omega)$$

$$\mathbf{h}_1^* \leftrightarrow \begin{cases} \delta[G_1(\omega)/|G_1(\omega)|], & \text{if } |G_1(\omega)| > \delta, \text{ for } 0 \le \omega \le \omega_1, \\ G_1(\omega), & \text{if } |G_1(\omega)| \le \delta, \text{ for } 0 \le \omega \le \omega_1, \\ G_1(\omega), & \text{elsewhere.} \end{cases} \tag{17}$$

These equations implicitly define the projectors P_2, and P_3, respectively.

Projection onto C_4: The projection of an arbitrary two-tuple $\mathbf{g}_0, \mathbf{g}_1) \in \mathbf{R}^N \times \mathbf{R}^N$ onto C_4 is again computed using the Lagrange multiplier method. The result is $(\mathbf{h}_0^*, \mathbf{h}_1^*) = P_4(\mathbf{g}_0, \mathbf{g}_1)$ where

$$h_0^*(n) = (-1)^n \frac{g_1(n) - g_1(N-n-1)}{2}$$

and

$$h_1^*(n) = (-1)^n h_0^*(n). \tag{18}$$

Finally, the VSPM algorithm takes the form

$$(g_0, g_1)_{k+1} = P_1 P_2 P_3 P_4 (g_0, g_1)_k. \tag{19}$$

Each projection is called a step. A new iteration cycle begins after four steps.

IV. Example and Numerical Results

In this experiment, we discretized the signals in the frequency domain to 512 samples. A linear phase QMF was designed with $N = 64$, $\sigma = 25 \times 10^{-4}$, $\delta = 3 \times 10^{-4}$, and a transition bandwidth $\Delta f = 0.172$. This is compared to the QMF's with the same Δf and N designed by Johnston's method in [5], where the author furnished the value of the impulse response. In order to demonstrate the value of VSPM for this problem, we chose the above (small) values of σ and δ that still allowed a nonempty intersection of all the constraints sets. To measure the error of the reconstructed signal for each filter, we applied an impulse $\delta(n)$ as input, and computed the energy of the residual error E of the output $y(n)$ i.e. $E = \sum_{n=0}^{2N}(y(n) - \delta(n-N-1))^2$.

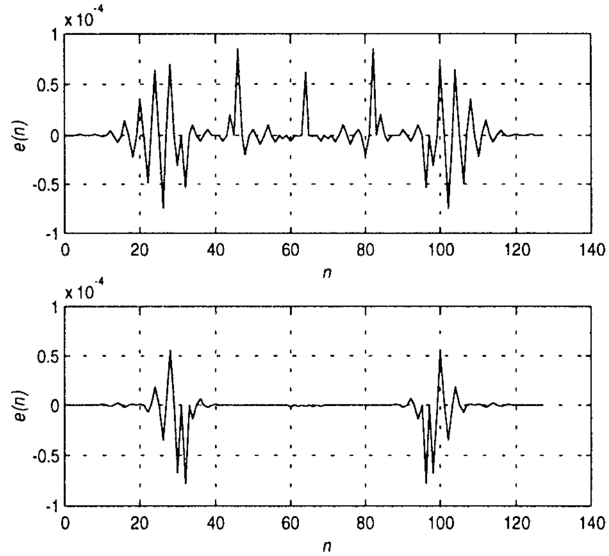

Fig. 4. Plot of the residual error $e(n)$ for the Johnston (top) and VSPM (bottom) pair of filters.

Table I shows that the QMF's designed by VSPM have superior stopband attenuation than the Johnston's design (71 dB versus 65 dB). In addition, the peak-to-peak deviation of the power complementary property of the VSPM designed filter is much smaller than that of the Johnston's filter and this led to smaller residual error E, as the last entry in Table I shows. From Fig. 2, we see that the VSPM's filter is equiripple.

The plot of the frequency responses of the Johnston and VSPM QMF's are shown in Fig. 2. Fig. 3 shows the plot of $|H_0(\omega)|^2 + |H_1(\omega)|^2$ in dB scale, and Fig. 4 shows the residual error $e(n) = y(n) - \delta(n-N-1)$. For readers who

TABLE I
CHARACTERISTICS AS DESIGNED BY VSPM AND THE JOHNSTON METHOD

	VSPM	Johnston
Stopband attenuation	71 dB	65 dB
$\|H_0(\omega)\|^2 + \|H_1(\omega)\|^2$ peak-to-peak deviation	0.005 dB	0.012 dB
Residual error E	1.77×10^{-4}	2.58×10^{-4}

TABLE II
FIRST HALF OF THE VSPM FILTER IMPULSE RESPONSE $\mathbf{H}_0$, THE SECOND HALF IS THE MIRROR IMAGE

n	VSPM QMFs COEFFICIENTS		n
1	0.00013088298316	-0.00043557700746	17
2	-0.00007452654723	0.00838931844286	18
3	-0.00003053811788	-0.00055024660557	19
4	0.00003987775353	-0.01255700407782	20
5	-0.00004002735530	0.00255927794332	21
6	0.00005470068848	0.01816875065335	22
7	0.00017690202124	-0.00619855026950	23
8	-0.00030653011763	-0.02582300936335	24
9	-0.00037654204409	0.01256117149338	25
10	0.00082208716903	0.03689109647512	26
11	0.00060271163836	-0.02412409348449	27
12	-0.00173048413854	-0.05535148773602	28
13	-0.00078208236958	0.04879669656944	29
14	0.00318063580290	0.09840378875785	30
15	0.00078938636202	-0.13643881722561	31
16	-0.00533870973317	-0.46134656497794	32

wish to compare these results with their own methods, we include in Table II the VSPM filter impulse response $\mathbf{h}_0$. The proposed VSPM algorithm for this example converged after about 25 000 cycles.

V. CONCLUDING REMARKS

We presented a new, promising, QMF design method based on vector space projection that allows more flexibility in the design in that any number of convex constraints can be incorporated in the design without the need to find one-step analytical solution. This method can be also easily extended to multidimensional QMF designs.

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