On the Relative Error Probabilities of Linear Multiuser Detectors*

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Abstract

The relative error-probability performance of three linear multiuser detectors - the minimum mean square error (MMSE) detector, the decorrelator, and the conventional matched filter (MF) detector - is investigated under nonorthogonal signaling and additive white Gaussian noise conditions. It is first shown that, somewhat surprisingly, for an arbitrary number of users with essentially arbitrary crosscorrelations, the MF detector outperforms both the MMSE and decorrelating detectors provided the power of the user of interest is sufficiently large. Subsequently the relative performance of the MMSE detector and the decorrelating detector is examined for the special case of two users. It is shown that there is a constant $\rho_\star \approx 0.9918$ such that, if the magnitude of the crosscorrelation of the two signature signals is less than $\rho_\star$, then the MMSE detector is superior to the decorrelating detector for any combination of signal and noise powers; whereas, if the crosscorrelation magnitude is larger than $\rho_\star$, then there exist combinations of signal and noise powers for which the decorrelating detector outperforms the MMSE detector. Finally for the two-user case we show that, under perfect power control, the MMSE detector always outperforms the decorrelating detector.

I Introduction and Background

Linear multiuser detection schemes have attracted considerable attention lately due to their simplicity, low complexity (as compared to optimum detection schemes) and satisfactory performance which, although not optimum in a minimum-error-rate sense, can nevertheless satisfy a number of alternative asymptotic optimization criteria such as high efficiency or near-far resistance.

The matched filter (MF) detector is, of course, the simplest linear detector. Since this detector neglects the presence of interfering users, its performance can be extremely poor in the presence of severe multiple access interference (MAI). Two key linear multiuser

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detectors are known to combat MAI very efficiently (see, for example, [3]). Specifically the decorrelating detector (or *decorrelator*) defeats MAI by selecting a proper linear filter to eliminate it, while the minimum mean square *error* (MMSE) detector achieves robustness against MAI by selecting the linear filter that minimizes the mean square value of the output MAI plus noise.

The MMSE detector has been the center of recent attention due to its noticeable feature of being practically implementable through blind adaptive schemes; that is, through schemes that use only the signature waveform of one user of interest and do not require training sequences or knowledge of signatures of interferers [1]. In [2] one can find a detailed analysis of this detector's performance under various conditions related to multiuser applications, along with efficient approximations of the corresponding error rates.

Several analytical and numerical results have suggested the conjecture (stated in [2]) that the MMSE detector outperforms the decorrelator for any combination of signal and noise powers. Furthermore a similar (if not stronger) feeling seems to be shared when the MMSE detector is compared against the conventional MF detector. It is the aim of this work to prove that both of these conjectures are in fact false. Specifically, for the comparison of the MMSE detector with the decorrelator we show that for the two-user case and for large enough crosscorrelation of the signature signals, it is possible to find noise and signal powers for which the MMSE detector is inferior to the decorrelator. We also show that it is not possible to find such cases under the condition of perfect power control. For the comparison of the MMSE detector against the MF detector we will show that, for the general K-user case with essentially arbitrary crosscorrelations, the MF detector outperforms the MMSE detector provided that the power of the user of interest is sufficiently large. We also show a similar result for the MF detector versus the decorrelator.

Let us now define the problem of interest more specifically. Consider a K-user binary communication system with corresponding normalized modulation waveforms $s_1, \ldots, s_K$, and signaling antipodally through an additive white Gaussian noise channel. If we limit ourselves to the synchronous signal case then the length-K vector y whose l-th component is the output of a filter matched to s_l is a sufficient statistic for the problem of detecting the transmitted symbols. The vector y can be written as [3, Page 56]

$$y = RA\bar{b} + \sigma n \tag{1}$$

where R denotes the normalized crosscorrelation matrix of the signals $s_1, \ldots, s_K$; A is the diagonal matrix $\mathrm{diag}\{A_1, \ldots, A_K\}$ with A_l the received amplitude of user-l; $\bar{b} = [b_1 \ldots b_K]^T$ is a vector whose l-th component is the symbol $b_l \in \{\pm 1\}$ of user-l; n is a $\mathcal{N}(0, R)$ random vector independent of $\bar{b}$, and finally σ^2 is the intensity of the additive channel noise.

As noted above we wish to examine the relative performance of the MMSE detector, the decorrelator and the MF detector. Without loss of generality throughout our analysis we will consider User # 1 to be the user of interest. The corresponding error probabilities are then given by the following formulas [3, Pages 113, 249, 300]:

$$P_{MMSE} = 2^{1-K} \sum_{b_2, \ldots, b_K \in \{-1,1\}^{K-1}} Q\left(\frac{e_1^T (R + X^{-2})^{-1} R X b}{\sqrt{e_1^T (R + X^{-2})^{-1} R (R + X^{-2})^{-1} e_1}}\right) \tag{2}$$

$$P_D = Q\left(x_1/\sqrt{e_1^T R^{-1} e_1}\right) \quad (3)$$

$$P_{MF} = 2^{1-K} \sum_{b_2,\ldots,b_K \in \{-1,1\}^{K-1}} Q\left(e_1^T R X b\right) \quad (4)$$

where $X = \mathrm{diag}\{x_1, \ldots, x_K\}$ with $x_l = A_l/\sigma$; $b = [1\ b_2 \cdots b_K]^T$; $e_1 = [1 0 \cdots . 0]^T$ and finally $Q(.)$ denotes the complementary unit cumulative Gaussian distribution. In the following sections, we compare these three expressions under various signaling conditions.

II The Matched Filter versus the MMSE and Decorrelating Detectors

In this section our goal is to show that *the MF outperfom both the MMSE and decorrelating detectors provided the user of interest is sufficiently strong in power.* In particular, we prove the following result.

Proposition 1: Fix $R, x_2, \ldots, x_K$, and assume that R is positive definite. If at least one interfering user has a signature waveform that is nonorthogonal to the signature waveform of User # 1, then there exists sufficiently large x_1 for which the MF detector outperforms the MMSE detector and the decorrelator.

Proof: The error probabilities for the detectors of interest are given in (2) - (4). It is convenient, however, to rewrite these expressions in order to reveal the linear dependency on x_1 of the arguments of the Q-functions. Since $X^{-2} = x_1^{-2} e_1 e_1^T + S$ with $S = \mathrm{diag}\{0, x_2^{-2}, \ldots, x_K^{-2}\}$ the matrix inversion lemma yields

$$(R + X^{-2})^{-1} e_1 = \frac{(R+S)^{-1} e_1}{1 + x_1^{-2} e_1^T (R+S)^{-1} e_1}. \quad (5)$$

Substituting this into the expression (2) for the error probability for the MMSE detector and noting that

$$e_1^T (R+S)^{-1} R e_1 = e_1^T (R+S)^{-1} (R + S - S) e_1 = 1 \quad (6)$$

(because $Se_1 = 0$), we obtain the following alternative expression

$$P_{MMSE} = 2^{1-K} \sum_{b_2,\ldots,b_K \in \{-1,1\}^{K-1}} Q\left(\frac{x_1 + e_1^T (R+S)^{-1} R \tilde{X} b}{\sqrt{e_1^T (R+S)^{-1} R (R+S)^{-1} e_1}}\right), \quad (7)$$

where $\tilde{X} = \mathrm{diag}\{0, x_2, \ldots, x_K\}$. It is also convenient to display explicitly the dependence of the MF detector's performance on x_1; namely,

$$P_{MF} = 2^{1-K} \sum_{b_2,\ldots,b_K \in \{-1,1\}^{K-1}} Q\left(x_1 + e_1^T R \tilde{X} b\right). \quad (8)$$

It should be noted that none of the quantities in bold face in (7) and (8) depends on x_1.

Using (6), the fact that $e_1^T R e_1 = 1$, and the Schwarz inequality, we can show that

$$\frac{1}{\sqrt{e_1^T (R+S)^{-1} R (R+S)^{-1} e_1}} \le 1, \quad (9)$$

with equality iff

$$R^{1/2}e_1 = \alpha R^{1/2}(R+S)^{-1}e_1 \tag{10}$$

for some scalar a. It is easy to verify that (10) holds iff $\rho_{l1} = 0$, $l = 2,\ldots,K$, where ρ_{l1} is the l-th component of the first column of $\boldsymbol{R}$. In other words we have equality in (9) iff simultaneously all interfering users have signature waveforms that are orthogonal to the signature waveform of User # 1. If at least one interfering user does not satisfy this constraint, then the inequality in (9) is strict.

Now, from (7) and (8), it follows immediately that, with (9) strict and for all sufficiently large x_1, we will have $P_{MMSE} > P_{MF}$. This is because each of these error probabilities is dominated in the tails (of large values of x_1) by the term involving the Q-function with the smallest argument. If (9) is strict, then for sufficiently large x_1 the smallest such argument of P_{MMSE} will be smaller than the smallest such argument in P_{MF}.

Similarly we can show the corresponding result for the decorrelator. Notice that using again the Schwarz inequality we can show

$$1 = e_1^T R^{-1/2} R^{1/2} e_1 \leq \sqrt{e_1^T R^{-1} e_1}\sqrt{e_1^T R e_1} = \sqrt{e_1^T R^{-1} e_1} \tag{11}$$

with equality iff $\alpha e_1 = Re_1$ for some scalar **a** or equivalently iff $\rho_{1l} = 0$, $l = 2\ldots,K$. The rest of the proof goes exactly as in the previous case. ■

For the case where the signatures of the interfering users are orthogonal to the signature of User # 1 we know that all three schemes have the same performance $P_{MMSE}(x_1) = P_D(x_1) = P_{MF}(x_1) = Q(x_1)$, which is of course the single-user performance.

III The MMSE Detector versus the Decorrelating Detector

We now turn, as in [2], to a comparison of the MMSE detector and the decorrelating detector. Here, we restrict attention to the two-user case (K = 2). Proceeding along the same lines of [2, Proposition 5.2] we will show the following result. If ρ denotes the crosscorrelation of the (normalized) signature waveformsof the two users, then the MMSE detector outperforms the decorrelator provided that $|\rho|$ is smaller than some upper limit $\rho_* < 1$. The significant new information brought by our result as compared to [2] (apart a slight improvement on the upper bound for $|\rho|$) is the fact that the proposed upper bound is in fact a tight one. By this we mean that *if $|\rho| > \rho_*$ then there is a combination of noise and signal powers for which the decorrelator outperforms the* ***MMSE*** *detector.*

It should be noted that, due to our last comment, we will be able to show that the conjecture stated in [2] (that the MMSE detector is always better than the decorrelator) is in fact false. At the end of this section we will also give an explicit counterexample to this conjecture.

To study this problem, it is convenient to rewrite the error probabilities for the two detectors using a slightly different notation. In particular, the error probabilities of interest can be written as follows

$$P_{MMSE} = \frac{1}{2}Q(ax_1 + b) + \frac{1}{2}Q(ax_1 - b) \tag{12}$$

and

$$P_D = Q(cx_1) \tag{13}$$

where

$$a = \frac{1-\rho^2+\phi_2}{\sqrt{(1+2\phi_2)(1-\rho^2)+\phi_2^2}}, \; b = a\frac{\rho\sqrt{\phi_2}}{1-\rho^2+\phi_2}, \; c = \sqrt{1-\rho^2}, \text{ and } \phi_2 = x_2^{-2}. \tag{14}$$

Notice that P_{MMSE} and P_D are symmetric in the correlation factor ρ so, without loss of generality, we will assume that $\rho \geq 0$. Let ϕ_2 and ρ be fixed. Then the parameters a, b, c also become fixed since they depend only on ϕ_2 and ρ and not on x_1. The two error probabilities can thus be written as functions of x_1, and we denote their difference by

$$J(x_1) = P_{MMSE}(x_1) - P_D(x_1). \tag{15}$$

In the following subsection we analyze this function in detail.

A. Analysis of $J(x_1)$

We first note that $J(0) = J(\infty) = 0$. Consider now the derivative of $J(x_1)$, which takes the form

$$J'(x_1) = \frac{1}{\sqrt{2\pi}}\left\{ce^{-c^2x_1^2/2} - \frac{1}{2}a\left[e^{-(ax_1+b)^2/2} + e^{-(ax_1-b)^2/2}\right]\right\}. \tag{16}$$

Before studying the sign of this derivative let us compare the two parameters a and c. From (14) it is easy to verify that a is strictly decreasing in ϕ_2. Since for $\phi_2 = 0$ the two quantities a and c become equal, we conclude that for any $\phi_2 > 0$ we have $a > c$.

To analyze the sign of the derivative $J'(x_1)$ it suffices to analyze the sign of the expression

$$\frac{\sqrt{2\pi}}{a}e^{c^2x^2/2}J'(x_1) = d - F(x_1) \tag{17}$$

where

$$F(x_1) = \frac{1}{2}\left\{e^{-(c_1x_1+c_2)^2/2} + e^{-(c_1x_1-c_2)^2/2}\right\}, \tag{18}$$

$$d = \frac{c}{a}e^{-b^2c^2/2(a^2-c^2)}, \; c_1 = \sqrt{a^2-c^2}, \text{ and } c_2 = ab/\sqrt{a^2-c^2}. \tag{19}$$

A typical form of the function $F(x_1)$ is depicted in Fig. 1. Notice that the number of sign alternations of the expression in (17) for $x_1 \geq 0$ depends on the number of positive real roots of the equation $d = F(x_1)$. This equation has at least one positive root (since the relation $J(0) = J(\infty) = 0$ guarantees one extremum for $J(x_1)$) and at most two (see Fig. 1). We can now prove the following lemma.

Lemma 1: Suppose ϕ_2 and ρ are fixed. Then the MMSE detector is better than the decorrelator for every value of x_1 iff $d \leq F(0)$.

Proof: If $d \leq F(0)$ then we note from Fig. 1 that the equation $d = F(x_1)$ has only one positive root (say x_*). This means that $J(x_1)$ is decreasing in x_1 for $0 \leq x_1 \leq x_*$ and increasing for $x_1 \geq x_*$. Since $J(0) = J(\infty) = 0$ this suggests that $J(x_1) \leq 0$; that is, that the MMSE detector is superior to the decorrelator for all values of x_1.

Conversely, if $d > F(0)$ then from Fig. 1 we observe that the MMSE detector is inferior to the decorrelator for values of x_1 sufficiently close to zero (since $J'(0) > 0$). ■

Let us now substitute all quantities entering in the inequality $d \leq F(0)$ in terms of the two parameters ρ and ϕ_2. We then obtain the following equivalent relation

$$G(\phi_2, \rho) = \frac{1 - \rho^2 + \phi_2}{\sqrt{1-\rho^2}\sqrt{(1+2\phi_2)(1-\rho^2) + \phi_2^2}} \, e^{\frac{-\rho^2 \phi_2}{2(1+2\phi_2)(1-\rho^2)+\phi_2^2}} \geq 1 \tag{20}$$

which due to Lemma 1 is necessary and sufficient for guaranteeing superior performance for the MMSE detector over the decorrelator for every value of x_1.

From the proof of Lemma 1 it is clear that if (20) is not satisfied for some combination of ϕ_2 and ρ, then we can then find small enough x_1 such that $P_{MMSE} > P_D$.

In order to proceed with our goal, namely to find all values of ρ for which the MMSE detector is better than the decorrelator for any value of x_1 and ϕ_2, we need to study the inequality in (20). This is the subject of the next subsection.

B. Analysis of $G(\phi_2, \rho)$

Let us fix ρ and consider $G(\phi_2, \rho)$ as a function of ϕ_2. We can then verify that the sign of its partial derivative with respect to ϕ_2 is the same as the sign of the following third order polynomial

$$U(\phi_2, \rho) = (1 - \rho^2 + \phi_2)^3 - \rho^2(1-\rho^2)(1-\rho^2+\phi_2) + 2\rho^2(1-\rho^2)^2. \tag{21}$$

Since the constant term of this polynomial is positive, the sign of the product of its three roots is negative. This in turn suggests that the polynomial has one real negative root and, depending on ρ, either two additional complex conjugate roots or two additional real roots that are of the same sign.

Using standard results concerning roots of third-order polynomials one can show that when $\rho > \rho_m = \sqrt{27/28} \approx 0.982$ the polynomial has three distinct real roots (one of which is negative), whereas for $0 < \rho \leq \rho_m$, the polynomial has one negative real root and two complex conjugate roots.

Consider the case $\rho > \rho_m$; as noted above, the polynomial $U(\phi_2, \rho)$ has three distinct real roots in this case, one of which is negative. We now need to identify the common sign of the two other real roots. If we substitute $\phi_2 = 2(1-\rho^2)$ then $U(2(1-\rho^2), \text{p}) < 0$ meaning that there is at least one root in the interval $(2(1-\rho^2), \infty)$, and consequently the remaining two roots are both positive. From the above we have the following remarks concerning the function $G(\phi_2, \text{p})$.

Remark 1: If $\rho \leq \rho_m$, then $U(\phi_2, \text{p})$ is positive for $\phi_2 \geq 0$, and consequently $G(\phi_2, \rho)$ is strictly increasing in ϕ_2 and larger than unity (since $G(0, \text{p}) = 1$). For this case the inequality in (20) is clearly satisfied for every $\phi_2 \geq 0$.

Remark 2: If $\rho > \rho_m$ then $G(\phi_2, \text{p})$, as a function of ϕ_2 and for $\phi_2 \geq 0$, presents two local extrema at the two positive real roots of $U(\phi_2, \text{p})$, the first one being a local maximum and the second a local minimum. Since the first extremum is a local maximum this means that its value is larger than unity (because $G(0, \text{p}) = 1$). The local minimum however can be either larger or smaller than unity depending on p. If for some ρ the corresponding local minimum of $G(\phi_2, \rho)$ goes below unity then inequality (20) is clearly not satisfied for every $\phi_2 \geq 0$. If on the other hand the local minimum is greater or equal than unity then (20) is satisfied for all $\phi_2 \geq 0$.

The variable behavior of $G(\phi_2, p)$ is depicted in Fig. 2 where we plot this function with respect to ϕ_2 and for characteristics values of the parameter p. We are now in a position to find all values of the crosscorrelation ρ for which the MMSE detector outperforms the decorrelator for all possible noise and signal powers.

Proposition 2: There exists a ρ_* with $\sqrt{27/28} < \rho_* < 1$ such that if $0 \leq \rho \leq \rho_*$ then the MMSE detector outperforms the decorrelator for any value of x_1 and ϕ_2; furthermore if $1 \geq \rho > \rho_*$ then there exist values for x_1 and ϕ_2 such that the MMSE detector is inferior to the decorrelator.

Proof: Due to Remark 1 we conclude that for any $0 \leq \rho \leq \rho_m$ we have that $G(\phi_2, \rho) \geq 1$ for all $\phi_2 \geq 0$. This of course implies that for any $\rho \in [0, \rho_m]$ the MMSE detector outperforms the decorrelator for any noise and signal powers. With the help of Remark 2 we will be able to slightly improve this result and propose an upper bound on ρ that is in fact tight.

Consider the case $1 \geq \rho > \rho_m$ where Remark 2 applies. If we plot the local minimum of $G(\phi_2, \rho)$ as a function of ρ then from Fig. 3 we have that this function is decreasing in p. Furthermore there exists a value ρ_* for which the corresponding local minimum is exactly equal to unity. Due to Remark2 and the monotonicity of the local minimum as a function of ρ we then conclude that for $\rho_* \geq \rho > \rho_m$ we have $G(\phi_2, p) \geq 1$ for all values of ϕ_2. On the other hand if $1 \geq \rho > p_{,,}$ there exist values for ϕ_2 where the inequality $G(\phi_2, p) \geq 1$ is false (see Fig. 2).

Combining the two intervals we conclude that for any $0 \leq \rho \leq \rho_*$ the inequality in (20) is true for all values of ϕ_2 therefore the MMSE detector outperforms the decorrelator for any noise and signal powers. The upper limit ρ_* can be computed numerically; the value we obtain is $\rho_* = 0.991765239964$.

From the above discussion we can also deduce that the proposed upper limit ρ_* is tight since for any $1 \geq \rho > \rho_*$ we can find values for the parameter ϕ_2 such that (20) is false, meaning that there exists sufficiently small x_1 such that $J(x_1) > 0$. ∎

From the analysis it is also apparent that if we had used x_2 instead of ϕ_2 this would have resulted in a function $G(x_2, \rho)$ with local extrema appearing at (very) large values of x_2 thus complicating the presentation.

C. Numerical (Counter) Example

A counter example for the conjecture in [2] is the following. Let $\rho = 0.995 > \rho_*$, from Fig. 2 we then observe that if we select for example $\phi_2 = 0.05$ then inequality (20) is false, indeed we have $G(0.05, 0.995) \approx 0.824 < 1$. According to our analysis there exists small enough x_1 such that the MMSE detector is inferior to the decorrelator. If we select x_1 even as large as $x_1 = 1$ this yields $P_{MMSE} = 0.4634 > P_D = 0.4602$.

D. Power Control

In this subsection we are going to consider the special case $A_1 = A_2 = A$ or $x_1 = x_2 = x$ ($\phi_2 = x^{-2}$) for the two-user case. The relative performance of the two detectors for this practically interesting case is analyzed in the next proposition.

Proposition 3: For the two-user case and under perfect power control the MMSE detector always outperforms the decorrelator.

Proof: Notice from (12) and (13) that for $\rho = 1$ and for $x_1 = x$, $\phi_2 = x^{-2}$ we have $P_{MMSE}(x) = P_D(x)$. Let us therefore consider the case $\rho < 1$. We can now apply the following change of variables $x = \xi/\sqrt{1-\rho^2}$ after which the corresponding error probabilities then become

$$P_{MMSE}(\xi,\rho) = \frac{1}{2}\sum_{b_2\in\{\pm 1\}} Q\left(\xi\frac{\xi^2+1+b_2\rho}{\sqrt{\xi^4+(1-\rho^2)(2\xi^2+1)}}\right), \tag{22}$$

and $P_D(\xi) = Q(\xi)$.

Our intention next is to show that $P_{MMSE}(\xi,\rho)$, for fixed ξ, is a decreasing function of p. This property will be sufficientto prove our proposition because we can then write $P_{MMSE}(\xi,\rho) \le P_{MMSE}(\xi,0) = Q(\xi) = P_D(\xi)$.

To show that the partial derivative of $P_{MMSE}(\xi,\rho)$ with respect to ρ is negative, after some tedious but straightforward calculations, can be shown to be equivalent to the following inequality

$$\rho\frac{2\xi^2+1}{\xi^2+1} \ge \tanh\left(\rho\frac{\xi^2(1+\xi^2)}{\xi^4+(1-\rho^2)(2\xi^2+1)}\right). \tag{23}$$

Using the fact that for any positive z we have $1 \ge \tanh(z)$, we can show that for $\rho \ge (\xi^2+1)/(2\xi^2+1)$ inequality (23) is true because

$$\rho\frac{2\xi^2+1}{\xi^2+1} \ge 1 \ge \tanh\left(\rho\frac{\xi^2(1+\xi^2)}{\xi^4+(1-\rho^2)(2\xi^2+1)}\right). \tag{24}$$

On the other hand for $\rho^2 \le (\xi^2+1)/(2\xi^2+1)$, and since for $z \ge 0$ we have $z \ge \tanh(z)$, the inequality in (23) is again true because

$$\rho\frac{2\xi^2+1}{\xi^2+1} \ge \rho \ge \tanh(\rho) \ge \tanh\left(\rho\frac{\xi^2(1+\xi^2)}{\xi^4+(1-\rho^2)(2\xi^2+1)}\right). \tag{25}$$

And we complete the proof by noting that the two intervals of ρ, where (23) was shown to be true, completely cover the interval [0 1] due to the fact that $(\xi^2+1)/(2\xi^2+1) \le 1$. ■

IV Conclusion

The results described in this paper have been pursued primarily out of theoretical interest. The significant practical advantages of the MMSE detector over the decorrelator and the matched filter would likely outweigh any performance disadvantage revealed here, inasmuch as the range of parameters for which the performance disadvantages arise are somewhat at the extremes for practical systems. Nevertheless, these results do provide some cautionary guidance concerning the relative merits of linear multiuser detectors.

It should be noted that further results can be obtained in this general area. For example, an interesting question arises as to the relative performance of the MF and MMSE detectors under the condition of perfect power control. One might conjecture that the MMSE detector would be superior in this case. However, counterexamples have been found that show this conjecture to be false in the cases of K = 3 and K = 5. These and additional related results will be included in future works on this subject.

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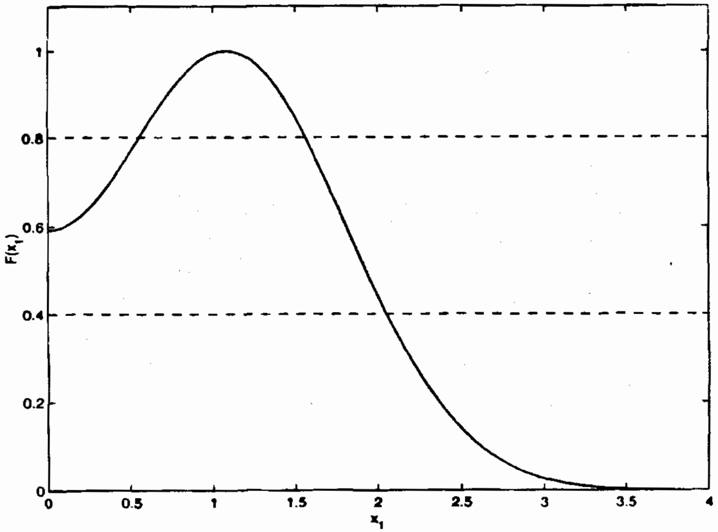

Figure 1: Typical form of the function $F(x_1)$

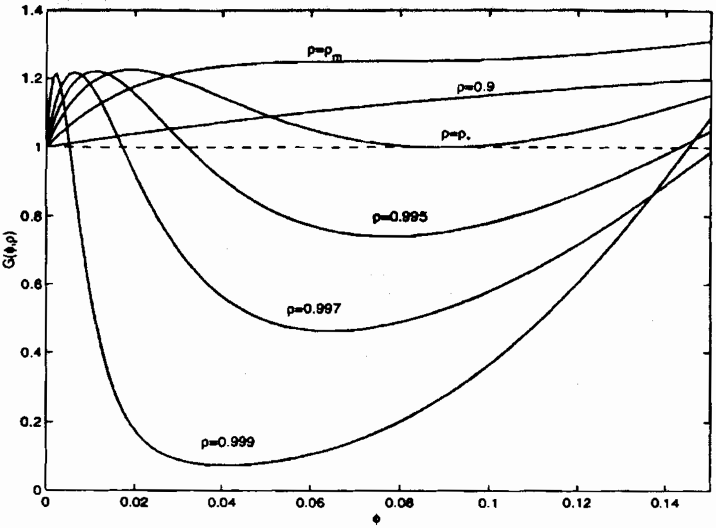

Figure 2: $G(\phi_2, \rho)$ as a function of ϕ_2 for various values of ρ.

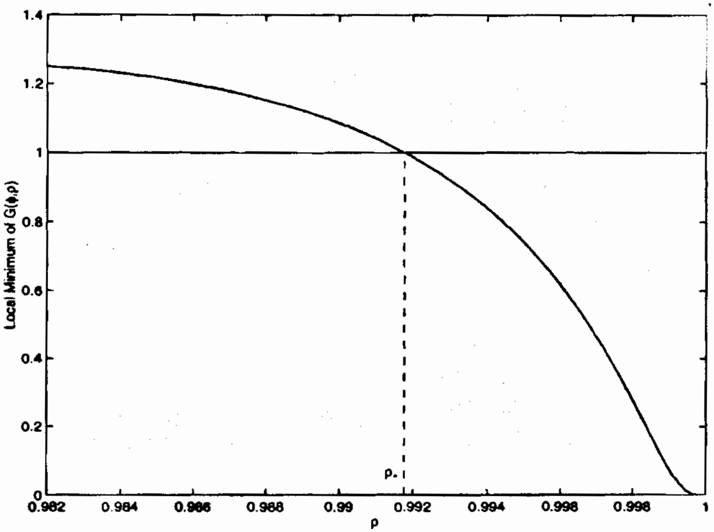

Figure 3: Local minimum of $G(\phi_2, \rho)$ as a function of ρ.