MIN - MAX DETECTION OF A WEAK SIGNAL IN STATIONARY MARKOV NOISE

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ABSTRACT

The detection of a constant weak signal in stationary Markov noise is considered . The observation sequence is passed through a ZNL and the sum of the outputs is compared to a fixed threshold. Using the efficacy as measure , the nonlinearity that leads to a min-max performance is obtained.

I. INTRODUCTION

Huber's ideas of robustness [1,2] are applied in [3,4 and 5] in order to obtain structures that are asymptotically robust signal detectors. It is assumed that the observation sequence is i.i.d. and that the common probability density belongs to an $\varepsilon-$ contamination class. In [3,4] the common density is also symmetric and the detection structure is a sum of zero memory nonlinear transformations of the observation sequence , compared to a fixed threshold. Using the efficacy as the performance criterion ,the robust structure turns out to be the canonical *limiter correlator* detector,when the signal is assumed small and the number of observations large. In [5] the assumption of symmetry is dropped ; instead symmetry within an interval around the origin is assumed. The same detection structure is used as in the other cases but here the threshold cannot be fixed if a desired level must be attained.

All of the above approaches assume that the observations are independent . In [6] the same detection problem is considered but the observations come from a moving average type process and are weakly dependent. In this paper we are dealing with dependent observations too but we are using a different model of dependency. It is assumed that the observations form a strictly stationary Markov sequence. As detection structure we use sums of memoryless nonlinear transformations and our goal is to optimize this structure. Obviously this structure is not the most optimal but it will give us an idea on how much the independence assumption structure changes under dependency and also if the performance changes drastically.

II. PRELIMINARIES .

Let $\{N_i\}$ be a strictly stationary Markov noise sequence. Since the statistics of such a sequence are well defined if we specify the bivariate distribution of two adjacent observations , we will assume that this bivariate distribution belongs to a class $F_{\varepsilon,m}$ which is defined as follows

$$f(x,y) = f(x)f(y)\{1+\lambda(x,y)\} \qquad (1)$$

$$\int_{-\infty}^{\infty}\lambda(x,y)f(x)dx = \int_{-\infty}^{\infty}\lambda(x,y)f(y)dy = 0 \quad (2)$$

$$f(x) = (1-\varepsilon)g(x)+\varepsilon h(x) \qquad 0 \le \varepsilon < 1 \quad (3)$$

$$|\lambda(x,y)| \le m < 1 \quad (4)$$

Condition (1) defines a common representation of a bivariate distribution. Condition (2) says that the function $f(x)$ is the marginal density. With Condition (3) we define an $\varepsilon-$ contamination model for the marginal. The density $g(x)$ is assumed to be known, symmetric, strongly unimodal and not equal to zero. The density $h(x)$ is unknown but symmetric and such that $f(x)$ has finite Fisher's information and the set $\{x : f'(x) = 0\}$ has $f-$ measure zero. Condition (4) limits our dependency model; notice that when $m = 0$ we are back to the i.i.d. case. The two constants ε and m are assumed to be known. As we will see shortly, Condition (4) is not only important because it makes $f(x,y)$ nonnegative but also because it makes the sequence $\{N_i\}$ a $\varphi-$ mixing sequence, a sufficient condition to guarantee asymptotic normality.

We now consider the detection of a constant signal; in particular, we would like to decide between the two hypotheses

$$H_0 : \qquad X_i = N_i$$
$$H_1 : \qquad X_i = N_i + s \qquad i = 1,2,\ldots$$

where $\{X_i\}$ is the observation sequence and s a signal that tends to zero. Adopting the terminology from [5], as our detection structure we will use the *nonlinear-correlator* (NC) detector, which is of the form

$$T_n(x) = \sum_{i=1}^{n}\psi(x_i) \quad (5)$$

For the performance criterion we will use the efficacy. In order for the efficacy to exist and to be a valid criterion we have to impose restrictions on the nonlinearity $\psi(x)$, which will determine the allowable class Ψ of nonlinearities $\psi(x)$. We assume that $\psi(x)$ is an odd symmetric, zero mean, and second order function such that

$$\sigma_0^2(\psi) = E\{\psi(N_1)^2\} + 2\sum_{j=1}^{\infty}E\{\psi(N_1)\psi(N_{j+1})\} > 0 \quad (6)$$

We also assume that $\psi(x)$ satisfies conditions that are sufficient for the validity of the Pittman - Noether theorem. Such conditions are given in [7]. The assumption that $\psi(x)$ is odd symmetric is reasonable since the marginal densities are assumed even symmetric functions.

Proposition 1. Let $\{N_i\}$ be a strictly stationary Markov sequence with bivariate density $f(x,y) \in F_{\varepsilon,m}$; also let $\psi(x) \in \Psi$. Then $\{N_i\}$ is a $\varphi-$mixing sequence and $\sigma_0^2(\psi)$ defined in (6) is absolutely summable.

Proof. Because the sequence is stationary Markov, the bivariate density between N_1 and N_{j+1} is given recursively by

$$f_j(n_1,n_{j+1}) = \int_{-\infty}^{\infty} \frac{f_{j-1}(n_1,n_j)f(n_j,n_{j+1})}{f(n_j)} dn_j \qquad j=2,3,\ldots \tag{7}$$

where $f_1(x,y) = f(x,y)$. Now using induction and Equations (1,2,4,7) we can show that the density defined by (7) has the following form

$$f_j(x,y) = f(x)f(y)\{1+\lambda_j(x,y)\} \qquad j=2,3,\ldots \tag{8}$$

where $\lambda_j(x,y)$ satisfies equation (2) and is given by

$$\lambda_j(x,y) = \int_{-\infty}^{\infty}\lambda_{j-1}(x,z)\lambda(z,y)f(z)dz \qquad j=2,3,\ldots \tag{9}$$

and we define $\lambda_1(x,y) = \lambda(x,y)$. Also we have that

$$|\lambda_j(x,y)| \le m^j \tag{10}$$

Now let A_1 be an event from the σ-algebra generated by $\{N_1,N_2,\ldots,N_k\}$ and A_2 an event from the σ-algebra generated by $\{N_{k+i},N_{k+i+1},\ldots\}$. Since the set $\{N_{k+1},\ldots,N_{k+i-1}\}$ is not involved in the event $A_1 \cap A_2$ we have that

$$P(A_1\cap A_2) = \int_{A_1\cap A_2} [\frac{f(n_1,n_2)f(n_2,n_3)\cdots f(n_{k-1},n_k)}{f(n_2)f(n_3)\cdots f(n_{k-1})}]\times [\frac{f_i(n_k,n_{k+i})}{f(n_k)f(n_{k+i})}]\times[\frac{f(n_{k+i},n_{k+i+1})\cdots}{f(n_{k+i+1})\cdots}]\, dn_1dn_2\cdots \tag{11}$$

Using (8,10) and noting that the first and the third term in (11) are the multivariate densities of $\{N_1,N_2,\ldots,N_k\}$ and $\{N_{k+i},\ldots\}$ respectively, we have

$$|P(A_1\cap A_2) - P(A_1)P(A_2)| \le m^i\, P(A_1)P(A_2) \tag{12}$$

So the sequence is symmetrically φ-mixing with $\varphi_i = m^i$. Clearly since $m<1$ we have that $\varphi_i \to 0$ as $i\to\infty$ and also that $\sum_{i=1}^{\infty}\varphi_i^{1/2} < \infty$. And this takes care of the first part of Proposition 1.

To prove now that $\sigma_G^2(\psi)$ is absolutely summable, we use Equations (8,10) and we have

$$\begin{aligned}\sigma_G^2(\psi) &= \int_{-\infty}^{\infty}\psi^2(x)f(x)dx + 2\sum_{j=1}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\psi(x)\psi(y)f_j(x,y)dxdy \\ &\le \int_{-\infty}^{\infty}\psi^2(x)f(x)dx + 2\sum_{j=1}^{\infty}m^j\,[\int_{-\infty}^{\infty}|\psi(x)|f(x)dx\,]^2 \\ &= \int_{-\infty}^{\infty}\psi^2(x)f(x)dx + \frac{2m}{1-m}[\int_{-\infty}^{\infty}|\psi(x)|f(x)dx\,]^2\end{aligned} \tag{13}$$

and , finally , using Schwartz inequality and that $\psi(x)$ is second order we have

$$\sigma_0^2(\psi) \le \frac{1+m}{1-m} \int_{-\infty}^{\infty} \psi^2(x) f(x)\, dx \ < \infty$$

and this proves the second part of Proposition 1.

Proposition 2 . Assume that $\{N_i\}$ is a noise sequence defined as above and that $E[\psi(N_1)] = 0$ and $E([\psi(N_1)]^2) < \infty$ where $\psi(x)$ is a measurable function. Then

$$\sum_{i=1}^{n} \psi(N_i)\, n^{-1/2} \xrightarrow{D} N\big(0, \sigma_0^2(\psi)\big) \tag{15}$$

Proof. It is a direct consequence of [8 , Theorem 21.1].

III. MIN - MAX DETECTION.

As our performance criterion we will be using the efficacy. Under the preceding assumptions it takes the following form

$$eff\,[\psi(x), f(x,y)] = \frac{[\int_{-\infty}^{\infty} \psi(x) f'(x)\, dx]^2}{\sigma_0^2(\psi)} \tag{16}$$

The problem we want to solve is the following. We would like to find a nonlinearity $\psi_r(x) \in \Psi$ and a density $f_r(x,y) \in F_{\epsilon,m}$ such that

$$\sup_{\psi(x) \in \Psi} \ \inf_{f(x,y) \in F_{\epsilon,m}} \ eff\,[\psi(x), f(x,y)] = eff\,[\psi_r(x), f_r(x,y)] \tag{17}$$

The first step is to try for a given $\psi(x)$ to minimize (16) over the density . Notice that the density depends on two functions, the marginal $f(x)$ and the function $\lambda(x,y)$. Since it makes no difference , we minimize over $\lambda(x,y)$ first. From (13) we have that

$$\sigma_0^2(\psi) \le \int_{-\infty}^{\infty} \psi^2(x) f(x)\, dx + \frac{2m}{1-m} [\int_{-\infty}^{\infty} |\psi(x)|\, f(x)\, dx]^2 \tag{18}$$

Equality is achieved when $\lambda(x,y)$ is given by

$$\bar{\lambda}(x,y) = m \ sn_\psi(x)\ sn_\psi(y) \tag{19}$$

where $sn_\psi(x)$ is an odd symmetric function which for $x > 0$ is given by

$$sn_\psi(x) = \begin{cases} 1 & \text{if} \quad \psi(x) > 0 \\ -1 & \text{if} \quad \psi(x) < 0 \\ 1\, or -1 & \text{if} \quad \psi(x) = 0 \end{cases} \tag{20}$$

Notice that the definition leads to a legitimate bivariate density since all conditions (1 to 4) are satisfied . Also applying (9) we have

$$\bar{\lambda}_j(x,y) = m^j \, sn_\psi(x) sn_\psi(y) \qquad j=1,2,\ldots \tag{21}$$

Thus if $\lambda(x,y)$ is given from (19) the efficacy becomes only a function of $\psi(x)$ and $f(x)$ and takes the form

$$eff^*[\psi(x),f(x)] = \frac{[\int_{-\infty}^{\infty}\psi(x)f'(x)dx]^2}{\int_{-\infty}^{\infty}\psi^2(x)f(x)dx + \frac{2m}{1-m}[\int_{-\infty}^{\infty}|\psi(x)|f(x)dx]^2} \tag{22}$$

In order to continue we have to minimize (22) over $f(x)$ and then maximize the result over $\psi(x)$. This is a min-max problem in itself with criterion function the eff^* given by (22). It turns out that this new problem has a saddle point ; in other words , there exists a pair $\psi_r(x)$ and $f_r(x)$ that satisfies the following saddle point relation

$$eff^*[\psi(x),f_r(x)] \le eff^*[\psi_r(x),f_r(x)] \le eff^*[\psi_r(x),f(x)] \tag{23}$$

for any allowable $\psi(x)$ and $f(x)$. As we know , any pair that satisfies (23) also satisfies a min-max relation like (17) , so the pair $\psi_r(x),f_r(x)$ is the one we are seeking. The left side inequality in (23) says that $\psi_r(x)$ is the optimum nonlinearity for $f_r(x)$, the one that will maximize the eff^* over the class Ψ The form of this optimum nonlinearity is given by the following theorem.

Theorem 1. Let $f(x)$ be a symmetric density function with finite Fisher's information and such that the set $\{ x : \psi_{lo}(x) = -\frac{f'(x)}{f(x)} = 0 \}$ has $f-$ measure zero . Then the nonlinearity $\psi_0(x)$ that maximizes (22) is given by

$$\psi_0(x) = \psi_{lo}(x) - \mu\varphi_0(x) \tag{24}$$

where $\varphi_0(x)$ is defined as

$$\varphi_0(x) = \begin{cases} \frac{1}{\mu}\psi_{lo}(x) & when \;\; -1 \le \frac{1}{\mu}\psi_{lo}(x) \le 1 \\ 1 & when \;\; 1 < \frac{1}{\mu}\psi_{lo}(x) \\ -1 & when \;\; -1 > \frac{1}{\mu}\psi_{lo}(x) \end{cases} \tag{25}$$

and μ is a constant that satisfies

$$S(\mu) = \mu + 2m \left[\frac{\int_{-\infty}^{\infty} f'(x)\varphi_0(x)dx}{1 - m + 2m\int_{-\infty}^{\infty}\varphi_0^2 f(x)dx} \right] = 0 \tag{26}$$

The proof of the theorem is in the Appendix. From Equations (24,25) we can see that the optimum nonlinearity $\psi_0(x)$ is closely related to the locally optimum $\psi_{lo}(x)$. Also the role of the function $\varphi_0(x)$ is to eliminate $\psi_{lo}(x)$ whenever it takes on values between $-\mu$ and μ. Now we are ready to define the pair that satisfies the saddle point. Since $\psi_r(x)$ is the optimum for $f_r(x)$ we need to define only $f_r(x)$

Theorem 2. The density $f_r(x)$ that gives the solution to the saddle point problem is the following

$$f_r(x) = \begin{cases} (1-\varepsilon)g(x_1)e^{x_1(x+x_1)} & \text{for } x \le -x_1 \\ (1-\varepsilon)g(x) & \text{for } |x| < x_1 \\ (1-\varepsilon)g(x_1)e^{-x_1(x-x_1)} & \text{for } x \ge x_1 \end{cases} \tag{27}$$

where $x_1 \ge 0$ and such that $f_r(x)$ has total mass equal to unity.

Proof. This density is nothing else than the one defined by Huber in [1,2]. It belongs to the $\varepsilon-$contamination class with a legitimate density $h(x)$ (see [1]). To find $\psi_r(x)$ we apply Theorem 1 and if we denote by $\psi_{lo}^r(x)$ the locally optimum nonlinearity of $f_r(x)$, we get

$$\psi_r(x) = \begin{cases} 0 & \text{for } 0 \le x \le x_2 \\ \psi_{lo}^r(x) - \psi_{lo}^r(x_2) & \text{for } x_2 \le x \le x_1 \\ \psi_{lo}^r(x_1) - \psi_{lo}^r(x_2) & \text{for } x \ge x_1 \end{cases} \tag{28}$$

For $x \le 0$ we recall that $\psi_r(x)$ is an odd function. The constant x_2 is defined as

$$\psi_{lo}^r(x_2) = \mu \tag{29}$$

and in order for (28) to be valid it has to satisfy $0 \le x_2 \le x_1$. In the Appendix we show that such an x_2 always exists.

Let us now for convenience define

$$M = \psi_{lo}^r(x_1) - \psi_{lo}^r(x_2) \tag{30}$$

Since $\psi_r(x)$ is nondecreasing we will have that $|\psi_r(x)| \le M$. Up to now we have shown that $\psi_r(x)$ and $f_r(x)$ satisfy the left inequality in (23). To prove the right one is straightforward. If we define as $n(f)$ and $d(f)$ the numerator and the denominator of the eff^* then

$$n(f) = \left[\int_{-\infty}^{\infty} \psi'_r(x) f(x)dx\right]^2 = \left[(1-\varepsilon)\int_{-\infty}^{\infty} \psi'_r(x) g(x)dx + \varepsilon \int_{-\infty}^{\infty} \psi'_r(x) h(x)dx\right]^2$$

$$\ge \left[(1-\varepsilon)\int_{-\infty}^{\infty} \psi'_r(x) g(x)dx\right]^2 = n(f_r) \tag{31}$$

and

$$d(f) = \int_{-\infty}^{\infty} \psi_r^2(x) f(x)dx + \frac{2m}{1-m}\left[\int_{-\infty}^{\infty} |\psi_r(x)| f(x)dx\right]^2$$

$$\leq (1-\varepsilon)\int_{-\infty}^{\infty}\psi_T(x)^2 g(x)dx + \varepsilon M^2 + \frac{2m}{(1-m)}\left[(1-\varepsilon)\int_{-\infty}^{\infty} \psi_T(x) g(x) dx + \varepsilon M\right]^2$$

$$= d(f_T) \tag{32}$$

From (31) and (32) we see that $f_T(x)$ simultaneously minimizes the numerator and maximizes the denominator of eff^*, so that the pair $\psi_T(x)$ and $f_T(x)$ satisfies also the right inequality in (23). This completes the proof.

Summarizing the results, the solution to our min - max problem is the following : the density $f_T(x,y)$ is given by

$$f_T(x,y) = f_T(x)f_T(y)\left\{1 + m\ sn_{\psi_T}(x)\ sn_{\psi_T}(y)\right\} \tag{33}$$

where $f_T(x)$ is defined in (27). The nonlinearity $\psi_T(x)$ is defined in (28) and these are all of the things we were looking for.

IV. NUMERICAL EXAMPLE.

Let the density $g(x)$ be the $N(0,1)$ normal. Then the $\psi_T(x)$ is

$$\psi_T(x) = \begin{cases} 0 & \text{for } |x| \leq x_2 \\ (|x|-x_2)sgn(x) & \text{for } x_2 \leq |x| \leq x_1 \\ (x_1-x_2)sgn(x) & \text{for } x_1 \leq |x| \end{cases} \tag{35}$$

For the density $f_T(x,y)$ we have the usual definition, only here the function $sn_{\psi_T}(x)$ can be equal to $sgn(x)$. In the following we give two tables, Table I contains the values of x_2 for different values of ε and m and Table II the ARE of $\psi_T(x)$ over $\psi_{lo}^T(x)$ when the underlying density is $f_T(x,y)$. Values for x_1 are not given. Since x_1 depends only on ε these values are the same with those given in [1] under the name k.

TABLE I

ε \ m	.1	.2	.3	.4	.5	.6	.7	.8	.9	1.0
.001	.147	.276	.400	.515	.635	.764	.912	1.096	1.373	2.630
.01	.145	.273	.393	.509	.626	.753	.895	1.069	1.320	1.945
.05	.139	.261	.374	.483	.591	.704	.826	.966	1.140	1.399
.10	.132	.247	.352	.452	.549	.648	.751	.861	.987	1.140
.15	.124	.232	.330	.422	.510	.597	.685	.776	.872	.980
.20	.117	.218	.309	.393	.472	.549	.625	.702	.780	.862
.30	.102	.190	.267	.337	.402	.463	.521	.577	.631	.685
.40	.087	.161	.226	.284	.337	.385	.430	.472	.512	.550
.50	.073	.134	.188	.235	.276	.314	.348	.380	.409	.436
.65	.051	.093	.130	.162	.190	.214	.236	.257	.274	.291
.80	.029	.054	.074	.091	.107	.121	.133	.143	.153	.162

TABLE II

ε \ m	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
0.001	1.01	1.03	1.07	1.12	1.19	1.29	1.42	1.64	2.04	4.48
0.01	1.01	1.03	1.06	1.11	1.18	1.26	1.38	1.55	1.85	2.68
0.05	1.01	1.03	1.05	1.09	1.14	1.21	1.29	1.39	1.54	1.77
0.1	1.01	1.02	1.05	1.08	1.12	1.17	1.22	1.29	1.38	1.49
0.15	1.00	1.02	1.04	1.07	1.10	1.13	1.18	1.23	1.28	1.35
0.2	1.00	1.01	1.03	1.06	1.08	1.11	1.14	1.18	1.22	1.27
0.3	1.00	1.01	1.03	1.04	1.06	1.08	1.10	1.12	1.14	1.16
0.4	1.00	1.01	1.02	1.03	1.04	1.05	1.06	1.08	1.09	1.1

APPENDIX.

Proof of Theorem 1. Notice that the value of the eff^* does not change if we multiply $\psi(x)$ by a constant. From one of the conditions for the validity of the Pittman - Noether theorem (Condition a) in [7]) we conclude that $\int_{-\infty}^{\infty}\psi(x)f'(x)dx < 0$. Thus maximizing Expression (22) is equivalent to maximizing

$$H(\psi) = -\int_{-\infty}^{\infty}\psi(x)f'(x)dx - \rho\left(\int_{-\infty}^{\infty}\psi^2(x)f(x)dx + \frac{2m}{1-m}\left[\int_{-\infty}^{\infty}|\psi(x)|f(x)dx\right]^2\right) \quad (36)$$

where ρ is a Lagrange multiplier. We will show that (36) is maximized by

$$\psi_0(x) = \frac{1}{2\rho}\left[-\frac{f'(x)}{f(x)} - \mu\varphi_0(x)\right] \quad (37)$$

where μ and $\varphi_0(x)$ are defined in Equations (25,26). Let now $\psi_1(x)$ be any other nonlinearity from the class Ψ. Define the following variation

$$J(\gamma) = -\int_{-\infty}^{\infty}[(1-\gamma)\psi_0(x) + \gamma\psi_1(x)]f'(x)dx - \rho\left(\int_{-\infty}^{\infty}[(1-\gamma)\psi_0(x) + \gamma\psi_1(x)]^2 f(x)dx\right.$$

$$\left. + \frac{2m}{1-m}\left[\int_{-\infty}^{\infty}\{(1-\gamma)|\psi_0(x)| + \gamma|\psi_1(x)|\}f(x)dx\right]^2\right) \quad (38)$$

where $\gamma \mathcal{E} [0,1]$. Notice that $J(0) = H(\psi_0(x))$ and $J(1) = H(\psi_1(x))$. By manipulating (38) we can rewrite it as

$$J(\gamma) - J(0) =$$

$$\gamma\int_{-\infty}^{\infty}\left[-f'(x) - 2\rho\psi_0(x)f(x)dx - \frac{4\rho m}{1-m}\left(\int_{-\infty}^{\infty}|\psi_0(x)|f(x)dx\right)\varphi_0(x)f(x)\right][\psi_1(x)-\psi_0(x)]dx$$

$$-\gamma \frac{4\rho m}{1-m} \left(\int_{-\infty}^{\infty} |\psi_0(x)| f(x) dx \right) \left[\int_{-\infty}^{\infty} [|\psi_1(x)| - |\psi_0(x)|] f(x) dx \right.$$

$$\left. - \int_{-\infty}^{\infty} \varphi_0(x) [\psi_1(x) - \psi_0(x)] f(x) dx \right]$$

$$-\gamma^2 \rho \left[\int_{-\infty}^{\infty} [\psi_1(x) - \psi_0(x)]^2 f(x) + \frac{2m}{1-m} \left\{ \int_{-\infty}^{\infty} [|\psi_1(x)| - |\psi_0(x)|] f(x) dx \right\}^2 \right] \quad (39)$$

It is enough to show that $J(\gamma) - J(0) \leq 0$.

From the way that $\varphi_0(x)$ is defined in (25) we can see that

$$|\psi_0(x)| = \psi_0(x) \varphi_0(x) \quad (40)$$

$$|\varphi_0(x)| \leq 1 \quad (41)$$

By multiplying (37) by $\varphi_0(x) f(x)$ and integrating we can show , using (26) and (40) , that

$$\frac{2m}{1-m} [\int_{-\infty}^{\infty} |\psi_0(x)| f(x) dx] = \frac{\mu}{2\rho} \quad (42)$$

If we substitute (42) in the first term of the difference $J(\gamma) - J(0)$ and also use (37) , we get zero. The second term using (40) becomes

$$-\gamma \frac{4\rho m}{1-m} \left(\int_{-\infty}^{\infty} |\psi_0(x)| f(x) dx \right) \left[\int_{-\infty}^{\infty} \left[|\psi_1(x)| - \varphi_0(x) \psi_1(x) \right] f(x) dx \right] \quad (43)$$

and because of (41) the quantity in the brackets is nonnegative. Thus for $\rho > 0$, the above expression becomes nonpositive. The third term for $\rho > 0$ is clearly nonpositive too. And we have that the difference $J(\gamma) - J(0)$ is nonpositive and in particular $J(1) \leq J(0)$. If we also define $\rho = ½$, Equation (37) becomes the same as (24).

For the existence of a μ that satisfies (26) notice the following : since by assumption $\psi_{lo}(x) \neq 0$ except on sets of $f-$ measure zero we have that as $\mu \to 0$ then $\frac{1}{\mu} \psi_{lo}(x) \to \pm\infty$ and thus $\varphi_0(x) \to sgn(\psi_{lo}(x)) = -sgn(f'(x))$. Thus from (26) we get

$$S(0) = -\frac{4m}{1+m} \int_{-\infty}^{\infty} |f'(x)| dx \leq 0 \quad (44)$$

Using Schwartz inequality it is easy to show that the integral in (44) is bounded by Fisher's information. As $\mu \to \infty$ the second term in $S(\mu)$ remains bounded so that $S(\mu) \to +\infty$. By continuity , there exist a μ that satisfies

$S(\mu) = 0$. And this completes the proof of Theorem 1.

Existence of x_2 . To prove that there exists an x_2 that satisfies $x_2 \leq x_1$ for the density defined in (27) it is enough to show that $S(\psi_{lo}^T(x_1)) \geq 0$. Notice that when $\mu = \psi_{lo}^T(x_1)$ then

$$\varphi_0(x) = \frac{\psi_{lo}^T(x)}{\psi_{lo}^T(x_1)} \tag{45}$$

On applying this in the expression for $S(\mu)$ and manipulating we get

$$S(\psi_{lo}^T(x_1)) = \psi_{lo}^T(x_1) \left[1 - \frac{2mFI}{[\psi_{lo}^T(x_1)]^2 (1-m) + 2mFI} \right] \geq 0 \tag{46}$$

where by FI we denote Fisher's information. Thus the solution to $S(\mu) = 0$ must be less than $\psi_{lo}^T(x_1)$ and since $\psi_{lo}^T(x)$ is a nondecreasing function we get $x_2 \leq x_1$. And this completes the proof.

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REFERENCES .

[1] P.J. Huber , " Robust estimation of a location parameter " *Ann. Math. Stat.*, vol 35 , pp. 73-101 ,1964.

[2] P.J. Huber , " A robust version of the probability ratio test," *Ann. Math. Stat.*, vol 36 , pp. 1753-1758 , 1965 .

[3] R.D. Martin and S.C. Schwartz , " Robust detection of a known signal in nearly Gaussian noise," *IEEE Trans. Inf. Theory* , vol. IT-17 , pp. 50-56 , Jan. 1971 .

[4] S.A. Kassam and J.B. Thomas , " Asymptotically robust detection of a known signal in contaminated non-Gaussian noise," *IEEE Trans. Inf. Theory* , vol. IT-22 , pp. 22-26 , Jan. 1976.

[5] S.A. Kassam , G.V. Moustakides and J.G. Shin , " Robust detection of known signals in asymmetric noise," *IEEE Trans. Inf. Theory* , vol. IT-28 , pp. 84-91, Jan. 1982.

[6] H.V. Poor , " Signal detection in the presence of weakly dependent noise , Parts I and II," *IEEE Trans. Inf. Theory* , vol IT-28 , pp. 735-752 , Sept. 1982.

[7] D.R. Halverson and G.L. Wise , " Discrete time detection in $\varphi-$ mixing noise," *IEEE Trans. Inf. Theory* , vol. IT-26 , pp. 189-198 , March 1980.

[8] J. Billingsley , Convergence of probability measures. New York : Wiley , 1968, pp. 166-195.