MINIMAX ROBUST EQUALIZATION FOR RANDOM SIGNALS THROUGH UNCERTAIN CHANNELS†

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ABSTRACT

The problem of fixed filter design for equalization of an uncertain channel is considered. A robust filter is derived by seeking the minimax filter for channel characteristics defined by bounds on amplitude and phase functions for the mean-squared-error performance criterion. An application of the results in equalization of a multipath channel is considered, and numerical values are given for the performance of the robust filter.

I. INTRODUCTION

The use of some form of channel equalization is mandatory in many transmission schemes where the channel characteristics cannot be assumed to be ideal, that is, where linear amplitude and phase distortion is present. In many situations the channel characteristics are not simply non-ideal but are in general not exactly known and may be time-varying. One approach which has been widely applied in equalization in such cases is that of adaptive schemes. In some cases, however, the use of adapative equalizers may be impractical because of cost and complexity as well as speed-of-convergence requirements.

In this paper we apply the minimax formulation to the problem of fixed equalizer design for uncertain channel characteristics. Let $G(\omega)$ be the frequency response of the linear channel, and assume that a random signal with power spectral density $S(\omega)$ is transmitted through the channel and observed in additive noise with power spectrum $N(\omega)$ at the output of the channel. The problem is to design an equalizer with some frequency response $H(\omega)$ such that the output is a good estimate, in the mean-squared-error sense, of the transmitted signal. In the minimax approach we define a class of possible channel characteristics, and seek the equalizer $H_R(\omega)$ which minimizes its worst-case performance, in terms of mean-squared-error, over the class of channel characteristics. The equalizer $H_R(\omega)$ is thus the minimax (robust) equalizer.

We consider the specific class of channel characteristics defined by upper and lower bounding functions for the amplitude characteristic $|G(\omega)|$ and with the phase characteristics arg $G(\omega)$ lying in subsets $\Phi(\omega)$ of $[-\pi,\pi]$ (as a function of ω). The solution $H_R(\omega)$ for this particular class is obtained explicitly, and is shown to have a very interesting interpretation. At frequencies for which the signal-to-noise is high relative to channel uncertainties, the robust equalizer simply inverts the "average" channel. Otherwise, a Wiener-filter type of equalizer is obtained. We also present a numerical example to illustrate our results. The example considers an ideal transmission path and a secondary path with inexactly known amplitude and time-delay. The signal has a first-order low-pass spectrum and the noise is white for our example.

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It is interesting to note that the problem of fixed equalizer design for an ensemble of random channels was considered in [1], using a statistical approach rather than the minimax approach. A slightly more general cost function was utilized in [1], incorporating amplitude scaling. The results in [1] are, nonetheless, of the same type as those obtained here. Further work will include use of the amplitude-scaled cost function in our formulation, as well as a consideration of uncertainties in signal and noise spectra.

II. PROBLEM DEFINITION

The observed waveform at the channel output may be written as $(g*s)(t) + n(t)$, where g is the channel impulse response with corresponding frequency response $G(\omega)$ and $s(t)$, $n(t)$ are the input random signal and additive noise with zero means and respective power spectral densities $S(\omega)$ and $N(\omega)$. The signal and noise are assumed to be uncorrelated and their power spectra are assumed to be known. Let a filter with frequency response $H(\omega)$ be used at the channel output to obtain an estimate of the signal waveform. Then the mean-squared-error $e(G,H)$ between the signal $s(t)$ and its estimate can be easily shown to be

$$e(G,H) = \frac{1}{2\pi}\int_{-\infty}^{\infty} |1-G(\omega)H(\omega)|^2 S(\omega) + |H(\omega)|^2 N(\omega) \quad d\omega \tag{1}$$

We now assume that the channel frequency response is not precisely known, but that its amplitude characteristic is bounded by known functions A_L, A_U, so that

$$A_L(\omega) \leq |G(\omega)| \leq A_U(\omega) \tag{2}$$

Furthermore, the argument $\phi(\omega)$ of $G(\omega) = |G(\omega)|\exp[j\phi(\omega)]$ is assumed to lie in a known closed subset $\Phi(\omega)$ of $[-\pi,\pi]$ for each ω. Let the class of allowable frequency responses G so defined be $\mathcal{G}$. Our objective is to find the filter frequency response $H_R(\omega)$ which will satisfy

$$\min_{H} \max_{G\in\mathcal{G}} e(G,H) = \max_{G\in\mathcal{G}} e(G,H_R), \tag{3}$$

so that H_R will be the frequency response of a minimax robust filter which optimizes worst-case estimation performance, chosen from the class of all linear filters.

III. SOLUTION FOR MINIMAX FILTER

In (1), we first fix H and obtain max $e(G,H)$. To do this we need to maximize pointwise the term $|1 - G(\omega)H(\omega)|^2$ in the integrand in (1). Now this can be written as

$$|1-G(\omega)H(\omega)|^2 = 1 - 2|G(\omega)||H(\omega)|\cos(\phi(\omega) + \theta(\omega)) + |G(\omega)|^2 |H(\omega)|^2 \tag{4}$$

where $H(\omega) = |H(\omega)| \exp[j\theta(\omega)]$.

For any value of $|G(\omega)|$, this is maximized by picking $\phi(\omega) \in \Phi(\omega)$ to minimize $\cos[\phi(\omega) + \theta(\omega)]$. This minimizing value $\phi_W(\omega)$ of $\phi(\omega)$ is given as $\arg \min_{\phi \in \Phi} |(\phi+\theta) \bmod (2\pi) - \pi|$. We now consider maximization with respect to $|G|$[1]. Note that $1-2|G||H|\cos(\phi_W + \theta) + |G|^2 |H|^2$ is quadratic in $|G|$ and has a minimum at $|G| = \cos(\phi_W+\theta)/|H|$. It is therefore clear that the maximizing value $|G|_W$ of $|G|$ is

$$|G(\omega)|_W = \begin{cases} A_L(\omega), & \cos[\phi_W(\omega) + \theta(\omega)]/|H(\omega)| \geq [A_L(\omega)+A_U(\omega)]/2 \\ A_U(\omega), & \text{otherwise} \end{cases} \quad (5)$$

We now consider the minimization with respect to H of $e(G_W,H)$, where $G_W = |G|_W \exp[j\phi_W]$. First consider minimization with respect to the phase $\theta(\omega)$. We interpret $\Phi(\omega)$ as some set of points on the unit circle in the complex plane, and let $2\alpha(\omega)$ be the largest angle subtended (in radians) at the origin by any arc on the unit circle not in $\Phi(\omega)$. Let $\beta(\omega)$ be the angular location of the middle of this arc (see Figure 1). Then the filter phase $\theta(\omega) = \theta_R(\omega)$ minimizing $e(G_W, |H| \exp[j\theta])$ is

$$\theta_R(\omega) = \pi - \beta(\omega), \quad (6)$$

or that phase which when added to $\beta(\omega)$ transforms this point to the point π on the unit circle. With this phase angle we have

$$e(G_W, |H| \exp[j\theta_R]) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \{1 + 2\,|G|_W |H| \cos\alpha + |G|_W^2\, |H|^2\} S + |H|^2 N \quad d\omega \quad (7)$$

The integrand in (7) can be written as

$$I = S + (2S|G|_W \cos\alpha)|H| + (|G|_W^2 S+N)|H|^2 \quad (8)$$

which is a quadratic in $|H|$. Note that in this expression $|G|_W = A_L$ when $|H|$ is less then or equal to $-2\cos\alpha/(A_L+A_U)$, and is A_U otherwise. Note that for $|H| = 0$ or $-2\cos\alpha/[A_L+A_U]$, I has the same value for both $|G|_W = A_L$ and A_U. It is also clear that when $\cos\alpha$ is non-negative, the minimizing value

[1]The argument ω is dropped henceforth wherever clarity is not affected.

$|H| = |H|_R$ is $|H_R| = 0$. We therefore consider the case where $\cos\alpha < 0$. Now if we put $|G|_w = A_U$ in (8), the resulting function $I_U(|H|)$ has a minimum at a value of $|H|$ less than or equal to $-2\cos\alpha/(A_L+A_U)$, at which point I of (8) has $|G|_w = A_L$. This follows from the fact that the minimum for $I_U(|H|)$ occurs at $|H|^* = -SA_U\cos\alpha/(A_U^2 S + N)$ so that $1/|H|^* \geq -A_U/\cos\alpha \geq -(A_U+A_L)/2\cos$ Let $I_L(|H|)$ be I of (8) as a function of $|H|$ when $|G|_w = A_L$. Then depending on whether $I(|H|)$ is described by Figure 2(a) or 2(b), we finally get that the minimizing value $|H| = |H|_R$ of I is

$$|H_R(\omega)| = \begin{cases} 0, & \cos[\alpha(\omega)] \geq 0 \\ \dfrac{-2\cos[\alpha(\omega)]}{A_L(\omega) + A_U(\omega)}, & \dfrac{S(\omega)}{N(\omega)} \geq \dfrac{2}{A_L(\omega)[A_U(\omega) - A_L(\omega)]} \text{ and } \cos[\alpha(\omega)] < 0 \\ \dfrac{-S(\omega)A_L(\omega)\cos[\alpha(\omega)]}{A_L^2(\omega)S(\omega) + N(\omega)}, & \text{otherwise} \end{cases} \qquad (9)$$

Note that (6) gives the phase of the minimax filter.

From the definition of $\alpha(\omega)$ we see that it is a measure of the degree to which channel phase is known. For $\alpha(\omega) \leq \pi/2$ the minimax equalizer produces zero output, the mean value of the desired signal, regardless of the degree to which the channel amplitude characteristic is known. Otherwise, note that the condition in (9) dictating the choice of filter amplitude is $N(\omega)/[A_L^2(\omega)S(\omega)] \leq \frac{1}{2}\{[A_U(\omega)/A_L(\omega)]-1\}$. The right side of this inequality is a measure of channel amplitude uncertainty, while the left side is the maximum noise-to-signal ratio at the channel output. Therefore when noise-to-signal ratio is not larger than the measure of channel uncertainty, the robust filter is essentially the inverse of the "average" channel. Otherwise noise suppression is more important and the filter behaves essentially as a Wiener filter for the channel with lowest gain.

IV. UNCERTAIN MULTIPATH CHANNELS - AN EXAMPLE

Consider the particular situation where the channel freuqency response is $G(\omega) = 1+a\exp(-j\omega\tau)$ where the secondary path attenuation 1/a and time delay τ are not precisely known. Let the noise be white with spectral density N_0 and let the signal spectrum be

$$S(\omega) = \frac{N_0}{0.1 + 0.0695\tau_0^2\omega^2}, \qquad (10)$$

where N_0 and τ_0 are known parameters. We model the multipath uncertainty by $0.2 \leq a \leq 0.8$ and $\tau \leq \tau_0$. Since τ may be zero, it it clear that the upper bound $A_U(\omega)$ for $|G(\omega)|$ is 1.8. To determine $A_L(\omega)$ we observe that for ω such that $\omega\tau_0 \geq \pi$ we can always find a value $\tau \leq \tau_0$ such that $\omega\tau = \pi$ so that $A_L(\omega) = 0.2$ for $\omega \geq \pi/\tau_0$. From the geometry in Figure 3, we find that $A_L(\omega)$ is in general given by

$$A_L(\omega) = \begin{cases} [1.04 + 0.4\cos(\omega\tau_0)]^{\frac{1}{2}}, & \omega\tau_0 \leq 101.5° \\ \sin(\omega\tau_0), & 101.5° \leq \omega\tau_0 \leq 143.1° \\ [1.64 + 1.6\cos(\omega\tau_0)]^{\frac{1}{2}}, & 143.1° \leq \omega\tau_0 \leq 180° \\ 0.2, & \omega\tau_0 \geq \pi \end{cases} \tag{11}$$

From the geometry in Figure 3, we can similarly derive the upper and lower bounds defining the uncertainty intervals $\Phi(\omega)$ for $\phi(\omega)$. Without utilizing the above amplitude constraints, that is, considering only extreme values of the phase $\phi(\omega)$, we obtain for $\phi_L(\omega)$ and $\phi_U(\omega)$ defining $\Phi(\omega) = [\phi_L(\omega), \phi_U(\omega)]$ the values

$$\phi_L(\omega) = \begin{cases} -\sin^{-1} \dfrac{0.8\sin(\omega\tau_0)}{[1.64 + 1.6\cos(\omega\tau_0)]^{\frac{1}{2}}}, & 0 \leq \omega\tau_0 \leq 143.1° \\ -53.1°, & \omega\tau_0 \geq 143.1° \end{cases} \tag{12}$$

and

$$\phi_U(\omega) = \begin{cases} 0, & 0 \leq \omega\tau_0 \leq \pi \\ -\sin^{-1} \dfrac{0.8\sin(\omega\tau_0)}{[1.64 + 1.6\cos(\omega\tau_0)]^{\frac{1}{2}}}, & \pi \leq \omega\tau_0 \leq 216.9° \\ 53.1°, & \omega\tau_0 \geq 216.9° \end{cases} \tag{13}$$

We now use the above separately derived bounds for amplitude and phase characteristics to define the channel uncertainty. Note that this process results in a somewhat larger uncertainty class than is defined by the original bounds on a and τ. The quantity $\cos[\alpha(\omega)]$ in the result (9) is negative for all ω. The point of intersection of the characteristic S/N and $2/A_L(A_L - A_U)$ is at $\omega\tau_0 \approx 2\pi/3$.

Thus we get for the minimax robust filter the characteristic

$$|H_R(\omega)| = \begin{cases} \dfrac{2\cos\{[\phi_L(\omega) + \phi_U(\omega)]/2\}}{A_L(\omega) + A_U(\omega)} , & |\omega\tau_0| \leq \dfrac{2\pi}{3} , \\[2ex] \dfrac{A_L(\omega)\,\cos\{[\alpha_L(\omega) + \phi_U(\omega)]/2\}}{A_L^{\,2}(\omega) + 0.1 + 0.0695(\omega\tau_0)^2} , & |\omega\tau_0| > \dfrac{2\pi}{3} , \end{cases} \tag{14}$$

where ϕ_L, ϕ_U, and A_L are given in (11)-(13) and $A_U = 1.8$. The phase of the robust filter is $\theta_R(\omega) = -[\phi_L(\omega) + \phi_U(\omega)]/2$.

Without the minimax approach one might base the equalizer filter design on some "nominal" channel characteristic. In particular, we may take here the nominal channel frequency response to be $G_N(\omega) = 1 + 0.5\exp(-j\omega\tau_0/2)$. Let the filter which is optimal for this nominal channel have frequency response $H_N(\omega)$. The amplitude characteristic of this filter is given by

$$|H_N(\omega)| = \frac{[1.25 + \cos(\omega\tau_0/2)]^{\frac{1}{2}}}{1.25 + \cos(\omega\tau_0/2) + 0.1 + 0.0695(\omega\tau_0)^2} \tag{15}$$

In Figure 4 we compare the amplitude characteristics of the robust and nominal filters.

Some explicit values of mean-squared-error obtained with the nominal and robust filters are as follows: $e(G_N,H_N) = 4.76\ (2N_0/\tau_0)$, $e(G_N,H_R) = 5.36\ (2N_0/\tau_0)$, $e(G_U,H_N) = 7.4\ (2N_0/\tau_0)$ and $e(G_U,H_R) = 6.94\ (2N_0/\tau_0)$. Here $G_U(\omega) = 1 + 0.8\exp(-j\omega\tau_0)$, i.e. the channel with the strongest secondary path and with maximum path difference. Note that the robust filter loses some performance under nominal conditions, but performs better under this deviation from nominal assumptions.

V. DISCUSSION AND CONCLUSION

The above numerical example provides an illustration of the practical application of the results of this paper. While the performance of the minimax robust filter in this case was not very much better than that of the nominal filter, it should be kept in mind that the multipath channel uncertainties involved only two unknown parameters, from which a larger class of channel frequency responses was derived. Thus the robust filter is designed for a larger uncertainty class than originally specified. In light of this, the performance of the robust filter for the "nominal" channel is quite good. It is possible to show significant gain in performance for the robust filter relative to the nominal filter for other channel characteristics chosen from this larger uncertainty class.

An important observation made in section III was that at frequencies for which the maximum noise-to-signal ratio is less than a measure of channel uncertainty, the minimax robust equalizer is essentially the inverse of the "average" channel. Here the "average" channel has frequency response which lies in the "center" of the bounded uncertainty class, and the measure of channel uncertainty is $[(A_U(\omega)/A_L(\omega))-1]/2$. Note that this measures only amplitude uncertainty. In [1] the problem of equalization of a random channel characterized by an ensemble of possible channel responses was considered. One basic conclusion there was also very similar to the above observation. For frequencies where the maximum noise-to-signal ratio is large relative to channel uncertainty noise rejection becomes important and a Wiener-filter type of behavior is indicated. At these frequencies the minimax equalizer characteristic depends on the signal and noise spectra. Thus an extension of this work should consider uncertainties for these spectra also.

REFERENCE

[1[F. J. Brophy, G.J. Foschini, and R.D. Gitlin, "A Compromize Equalizer Design Incorporating Performance Invariance", BSTJ, Vol. 52, pp. 1077-1095, Sept. 1973.

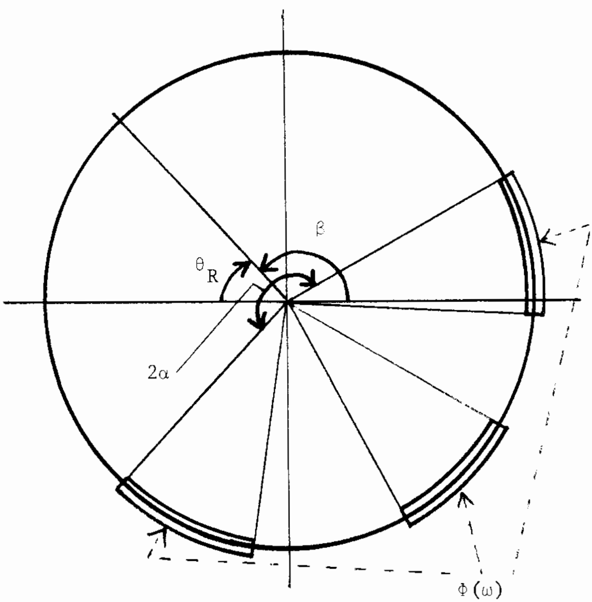

Figure 1. Illustration of the uncertainty set $\Phi(\omega)$ and the angles $\alpha(\omega)$, $\beta(\omega)$ and $\theta_R(\omega)$.

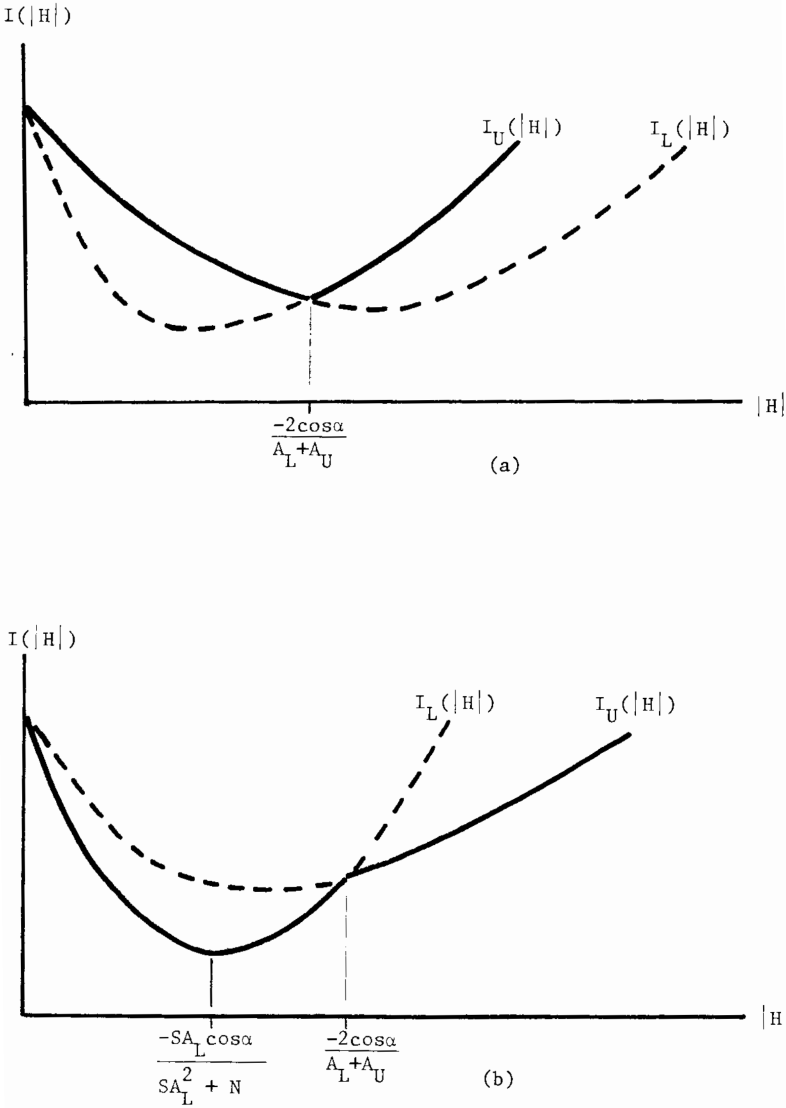

Figure 2. Minimization of I(|H|) [equation (8)] for solution of robust filter magnitude.

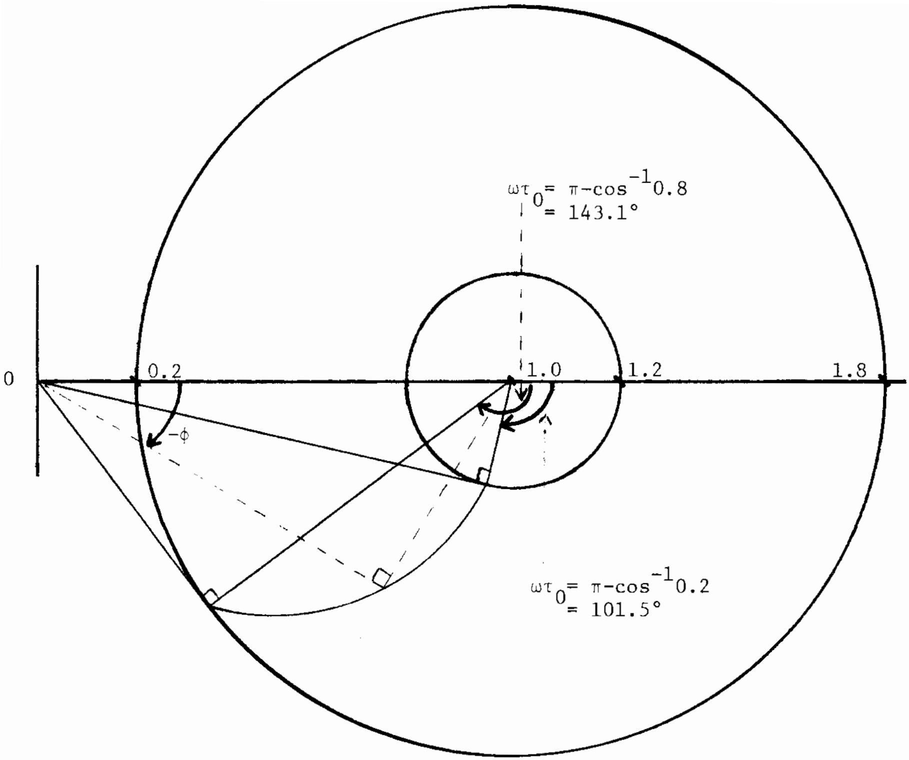

Figure 3. Geometry for computing amplitude and phase bounds for multipath channel example.

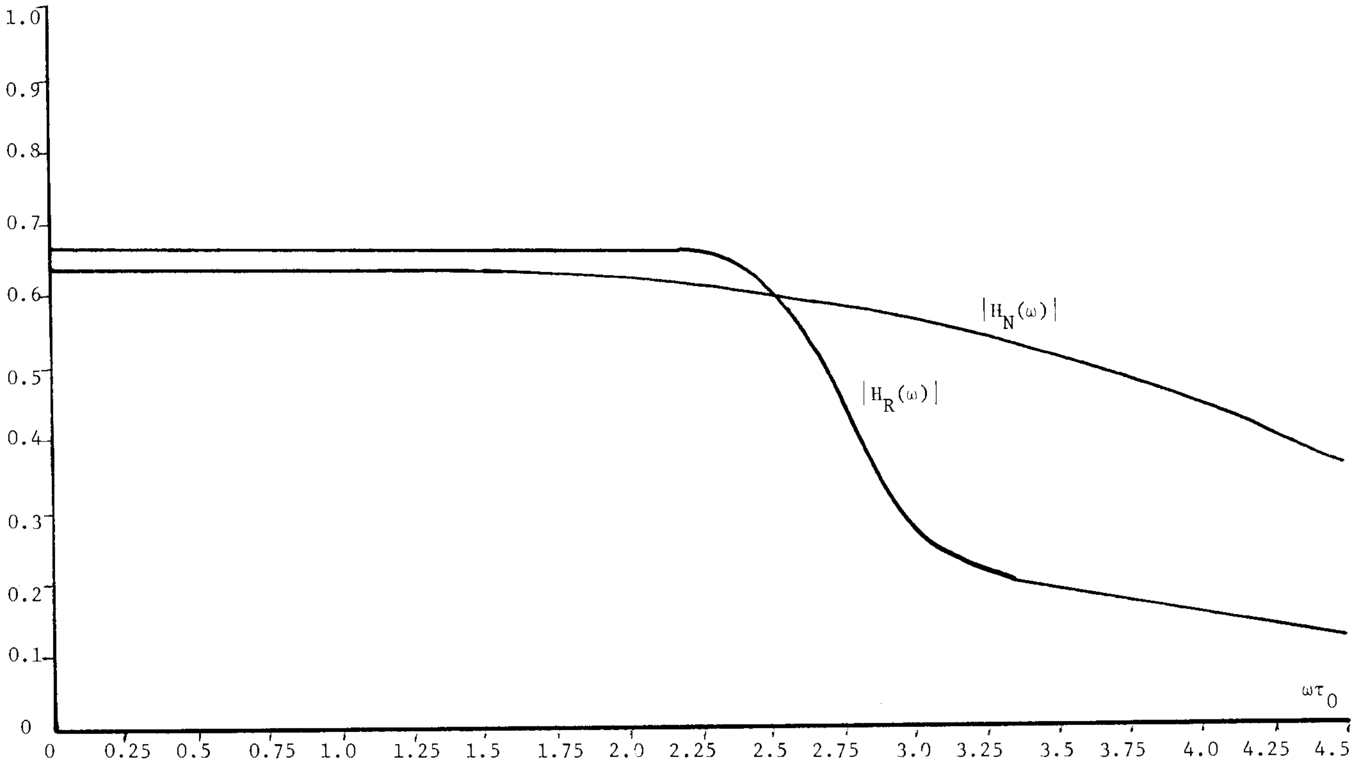

Figure 4. Magnitudes of nominal and robust filters for multipath example