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OPTIMAL POINT TARGET DETECTION WITH UNKNOWN PARAMETERS BY MIMO RADARS

Ali Tajer, Guido H. Jajamovich, and Xiaodong Wang
Electrical Engineering Department
Columbia University, New York, NY 10027
Email: {tajer, guido, wangx}@ee.columbia.edu

George V. Moustakides
Electrical Engineering Department
University of Patras, 26500 Rion, Greece
Email: moustaki@ece.upatras.gr

Abstract*—We consider multiple-input multiple-output (MIMO) radar systems with widely spaced transmit and receive antennas. We treat the problem of detecting point targets when one or more target parameters of interest are unknown. We provide a composite hypothesis testing framework for jointly estimating such parameters along with detecting the target while only a** finite **number of signal samples are available. The test offered is** optimal **in a Neyman-Pearson-**like **sense such that it provides a Bayesian-optimal detection test, minimizes the average mean-square parameter estimation error subject to an upper bound constraint on the false-alarm probability, and requires a finite number of samples. While the test can be applied for concurrently detecting the target along with estimating any unknown parameter of interest, we consider the problem of detecting a target which lies in an unknown space range and find the range through estimating the vector of time delays that the emitted waveforms undergo from being illuminated to the target until being observed by the receive antennas. We also analyze the** diversity gain **which we define as the rate that the probability of mis-detecting a target decays with the increasing SNR and show that for a MIMO radar system with N_t and N_r transmit and receive antennas, respectively, the diversity gain is 1 for point targets.

I. INTRODUCTION

In this paper we consider a widely-spaced antenna configuration and treat two problems. First, we analyze target detection when some radar parameters are unknown and we are interested in estimating them. We offer a framework for joint target detection and parameter estimation in a Neyman-Pearson-*like* sense (exact definition of optimality is provided in Section II-B) when the receive antennas can acquire only a *finite* number of observations. While the proposed framework can be exploited for detecting the target in conjunction with estimating any parameter of interest, we consider estimating the time-delays that the transmitted waveforms undergo before reaching the receive antennas. While such estimation is necessary for optimal detection, they also can be utilized in order to estimate the range of the target. We formulate this problem as a composite hypothesis test which is shown to be solved optimally via the widely-known generalized likelihood ratio test (GLRT). Note that the existing optimality results of the GLRT hold for the asymptote of an *infinite* number of observations under certain assumptions [1, Sec. 5.6].

As the second problem, we adopt the notion of diversity gain from MIMO communications and define it as the rate that the probability of target mis-detection decays with the increasing signal-to-noise ratio (SNR), while the false-alarm probability is kept below a certain level. We analytically assess the diversity gain yielded by MIMO radars as a function of the number of transmit and receive antennas. We treat the same problem for phased-array radars as well to furnish a benchmark for comparisons. The summary of the contributions and results of this work is as follows.

We propose an optimality measure which is shaped by target detection performance, parameter estimation accuracy, and false-alarm probability. The corresponding optimal composite hypothesis test which satisfies all the optimality criteria is introduced and is deployed for detecting a target in an unknown range. The range of the target is estimated via estimating time-delays. The existing literature on MIMO radars with widely-space antennas either embed the effect of the time-delays only as phase-shifts while ignoring the signal attenuation due to path-loss which is also a function of the time delay [2]–[4], or consider identical path-losses corresponding to different time-delays [5] and absorb it in the scatterer's reflectivity factor. We provide a model which considers different path-losses corresponding to different time-delays.

We characterize the maximum likelihood (ML) estimate of the time-delay vector which consists of N_tN_r components each corresponding to a pair of transmit-receive antennas. We also provide the optimal detector corresponding to the ML estimates found. Finally, we show that in an $N_t \times N_r$ MIMO radar configuration with widely-spaced antennas the mis-detection probability decays as SNR^{-1} for point targets.

II. System Descriptions

A. MIMO Radar

We consider a MIMO radar system comprising of N_t and N_r transmit and receive antennas, respectively, and adopt the classical Swerling case I model [6] extended for multiple-antenna systems [3], [4]. According to this model, a target consists of one or more small scatterers exhibiting random, independent and isotropic scintillation.

In this paper we consider *pint* targets and define the location of the target's scatterer in the Cartesian coordination by $X \triangleq (x, y, z)$. Also, we denote the reflectivity factor of the scatterer by ζ and assume that ζ is distributed as zero-mean complex random variables with variance 1, i.e., $\mathbb{E}[|\zeta|^2] = 1$. The target and the reflectivity factor are assumed to remain constant during a finite number of observations denoted by K and change to independent states afterwards.

Motivated by capturing the inherent diversity provided by independent scatterers, the antennas are spaced widely enough (such that they satisfy the conditions in [3, Sec. II.A]) to ensure having uncorrelated reflections from the target to the receive antennas. We assume that the transmit antennas are located at $X^t_m \triangleq (x^t_m, y^t_m, z^t_m)$, for $m = 1, \ldots, N_t$, and the receive antennas are located at $X^r_n \triangleq (x^r_n, y^r_n, z^r_n)$, for $n = 1, \ldots, N_r$. The transmit antennas emit N_t *narrowband* waveforms of duration T whose lowpass equivalents are given by $\sqrt{\frac{E}{N_t}} s_m(t)$ for $m = 1, \ldots, N_t$, where E is the total transmitted energy and $\int_T |s_m(t)|^2 dt = 1$. In contrast to the conventional phased-array radars which deploy waveforms which are identical upto a scaling factor [7], in MIMO radar systems these waveforms are designed such that they facilitate acquiring independent observations of each scatterer and often are assumed to be orthonormal, i.e.,

$$\int_T s_m(t) s^*_n(t)\, dt = \delta(m-n), \tag{1}$$

where * denotes complex conjugate and $\delta(\cdot)$ is the Dirac's delta function. The waveform illuminated by the m^{th} transmit antenna to the scatterer and received by the n^{th} receive antenna passes through an end-to-end distance which we denote by $d_{m,n}$ and undergoes a time delay which we denote by $\tau_{m,n} \triangleq d_{m,n}/c$, where c is the speed of light. By defining β as the path-loss exponent and superimposing the effects of all scatterers, the base-band equivalent of the signal received by the n^{th} receive antenna due to the waveform $s_m(t)$ transmitted by the m^{th} transmit antenna is given by

$$\begin{aligned} r_{m,n}(t) &= \sqrt{\frac{E}{N_t}} \zeta \Big(\frac{1}{d_{m,n}}\Big)^{\beta} s_m(t - \tau^p_{m,n}) e^{-j2\pi f_c \tau_{m,n}} \\ &+ z_{m,n}(t). \end{aligned} \tag{2}$$

Note that this model differs from those of [3] and [4] in the sense that we have added the attenuation effects of path-losses by including the terms $\Big(\frac{1}{d_{m,n}}\Big)^{\beta} = \Big(\tau_{m,n}\, c\Big)^{-\beta}$. The exponential term $\exp(-j2\pi f_c \tau_{m,n})$ in (2) represents the effect of propagation phase shift, where f_c is the carrier frequency, and $z_{m,n}(t) \sim \mathcal{CN}(0, \frac{1}{N_t})$, denotes the additive white Gaussian noise.

Using (2), the received signal at the n^{th} antenna, which is a superposition of all emitted waveforms, is given by

$$r_n(t) = \sqrt{\frac{E}{N_t}} \sum_{m=1}^{N_t} \frac{c^{-\beta}}{\tau^{\beta}_{m,n}} s_m(t - \tau_{m,n}) h_{m,n} + z_n(t),$$

where $h_{m,n} \triangleq \zeta\, e^{-j2\pi f_c \tau_{m,n}}$ and $z_n(t) \triangleq \sum_{m=1}^{N_t} z_{m,n}(t) \sim \mathcal{CN}(0,1)$. Furthermore, we assume that the waveforms are narrowband. Based on the narrow-band assumption, for $m = 1, \ldots, N_t$ and $n = 1, \ldots, N_r$ [4], we get

$$\forall \tau, \quad s_m(t) = e^{j2\pi f_c \tau}\, s_m(t-\tau), \tag{3}$$

which in conjunction with the orthonormality assumption (1) implies that

$$\forall\, \tau_{m,k}, \tau_{n,k},\ \int_T s_m(t - \tau_{m,k}) s^*_n(t - \tau_{n,k})\, dt = \delta(m-n). \tag{4}$$

We also define the time-delay vector $\boldsymbol{\tau} \triangleq [\tau_{1,1}, \ldots, \tau_{N_t,N_r}]$. Based on the model given in (2) and noting that the noise-terms are unit-variance, the transmission signal-to-noise ratio, denoted by SNR, is given by $\mathsf{SNR} = \frac{E}{T}$

B. Problem Statement

We assume that the receive antennas sample the received signal at the rate of $\frac{1}{T_s}$ samples per second. By defining $r_n[k] \triangleq r_n(kT_s)$, $z_n[k] \triangleq z_n(kT_s)$ and $s_m[k;\tau] \triangleq s_m(kT_s - \tau)$, the discrete-time low-pass equivalent of the received signal when a target is present is, for $k = 1, \ldots, K$, given by

$$r_n[k] = \sqrt{\frac{E}{N_t}}\, c^{-\beta} \sum_{m=1}^{N_t} \frac{1}{\tau^{\beta}_{m,n}} h_{m,n} s_m[k; \tau_{m,n}] + z_n[k]. \tag{5}$$

We also assume that the sampling rate is high enough to ensure that the discrete-time signals $s_m[k;\tau_{m,n}]$ remain orthogonal for arbitrary delays $\tau_{m,l}, \tau_{n,l}$, i.e., $\sum_k s_m[k;\tau_{m,l}] s^*_n[k;\tau_{n,l}] = \frac{1}{T_s}\delta(m-n)$. Let us define $\boldsymbol{r}[k] \triangleq [r_1[k], \ldots, r_{N_r}[k]]^T$ for $k = 1, \ldots, K$ and $\boldsymbol{R} \triangleq [\boldsymbol{r}[1]^T, \ldots, \boldsymbol{r}[K]^T]^T$. Also let $f_0(\boldsymbol{R})$ denote the probability density function (pdf) of the received signal when a target is not present. When a target is present, the pdf of the received signal depends on the unknown parameter $\boldsymbol{\tau}$ and is denoted by $f_1(\boldsymbol{R} \mid \boldsymbol{\tau})$. Therefore, by defining the estimate of $\boldsymbol{\tau}$ by $\hat{\boldsymbol{\tau}}$, the detection part of the problem can be cast as

$$\begin{cases} \mathcal{H}_0 : \text{No target exists at delay } \hat{\boldsymbol{\tau}} \text{ where } \quad \boldsymbol{R} \sim f_0(\boldsymbol{R}), \\ \mathcal{H}_1 : \text{Target exists at delay } \hat{\boldsymbol{\tau}} \text{ where } \quad \boldsymbol{R} \sim f_1(\boldsymbol{R} \mid \hat{\boldsymbol{\tau}}). \end{cases} \tag{6}$$

Our objective is to detect a target when the vector time-delays $\boldsymbol{\tau}$ is unknown such that that the following conditions are satisfied.

- C1) The average ML estimation error of the time delays $\boldsymbol{\tau}$ is minimized.
- C2) The false-alarm probability of the target detector is kept below a certain level.
- C3) For the given set of ML estimates $\hat{\boldsymbol{\tau}}$, the target detector is Bayesian-optimal, i.e., the Bayes risk is minimized [8, Sec. II.B].
- C4) The test requires only a finite number of samples, i.e., $K < \infty$.

We call the composite hypothesis test that satisfies all the conditions above Neyman-Pearson-*like* optimal as it follows the same spirit as the standard Neyman-Pearson (NP) criterion (minimize the detection error probability subject to a constraint on false-alarm probability).

Compared to the existing literature, e.g., [3] and references therein, in addition to the objectives (conditions C1-C4) and the model (including path-loss effects) our proposed framework has also a slightly different application. The existing models aim at detecting a target given that it lies within a given range, while in this paper we aim at estimating the range for the potentially existing target within any arbitrary subspace of the entire search space and detecting the presence of the target based on those estimates.

Note that when we are only interested in hypothesis testing (and not in estimating $\boldsymbol{\tau}$), the optimal test is given by

$$\frac{f_1(\boldsymbol{R})}{f_0(\boldsymbol{R})} = \frac{\int \pi(\boldsymbol{\tau}) f_1(\boldsymbol{R} \mid \boldsymbol{\tau}) d\boldsymbol{\tau}}{f_0(\boldsymbol{R})} \underset{\mathcal{H}_0}{\overset{\mathcal{H}_1}{\gtrless}} \lambda, \tag{7}$$

where λ is found by the conditions enforced on the tolerable level of false alarm. On the other hand, the optimal test that satisfies all the conditions C1-C4 given above is characterized by the following Theorem.

Theorem 1 (Moustakides [9]): For a finite cardinality vector $\boldsymbol{R} \in \mathbb{C}^{|\boldsymbol{R}|}$ and an unknown vector parameter $\boldsymbol{x}$, the optimal test in the NP-*like* sense (that satisfies C1-C4) for estimating $\boldsymbol{x}$ and deciding between $\mathcal{H}_0$ and $\mathcal{H}_1$ given as

$$\begin{cases} \mathcal{H}_0 : & \boldsymbol{R} \sim f_0(\boldsymbol{R}), \\ \mathcal{H}_1 : & \boldsymbol{R} \sim f_1(\boldsymbol{R} \mid \boldsymbol{x}), \end{cases} \tag{8}$$

is

$$\frac{f_1(\boldsymbol{R} \mid \hat{\boldsymbol{x}})}{f_0(\boldsymbol{R})} \underset{\mathcal{H}_0}{\overset{\mathcal{H}_1}{\gtrless}} \lambda, \quad \text{where} \quad \hat{\boldsymbol{x}} = \arg\max_{\boldsymbol{x}} \pi(\boldsymbol{x}) f_1(\boldsymbol{R} \mid \boldsymbol{x}), \tag{9}$$

and $\pi(\boldsymbol{x})$ is the prior distribution of $\boldsymbol{x}$.

In the theorem above, $\hat{\boldsymbol{x}}$ given in (9) is the *maximum a posteriori* (MAP) estimate of $\boldsymbol{x}$. As we do not have any prior information about the location of the target, throughout the analysis we assume that $\pi(\boldsymbol{\tau})$ has a uniform distribution. Hence, the MAP estimate of $\boldsymbol{\tau}$ becomes its ML estimate. The above Theorem essentially establishes the GLRT as the optimal estimation/detection strategy that satisfies the conditions C1-C4. Several other *asymptotic* optimality results are known for the GLRT which are all based on having an *infinite* number of observation [9] and [10, Chapter 22].

III. Joint Detection and Estimation for Point Targets

In this section we consider the application of MIMO radars for high-resolution detection in terms of detecting point-object targets or exposing single-scatterers. In this case, the target is modeled by one scatterer located at its gravity center X_0. Note that in (3) by setting $P = 1$ we get $h_{m,n} = \zeta \, e^{-j2\pi f_c \tau_{m,n}}$, where $\zeta \sim \mathcal{CN}(0, 1)$. Therefore, the null and alternative hypotheses are given respectively by

$$\begin{cases} \mathcal{H}_0 : & r_n[k] = z_n[k], \\ \mathcal{H}_1 : & r_n[k] = \sqrt{\frac{E}{N_t}} \zeta \sum_{m=1}^{N_t} \frac{e^{-j2\pi f_c \tau_{m,n}} c^{-\beta}}{\tau_{m,n}^{\beta}} \, s_m[k; \tau_{m,n}] \\ & \qquad + z_n[k]. \end{cases} \tag{10}$$

We are interested in solving the optimum test given in (9) and we start by determining the ML estimate of the delay vector $\boldsymbol{\tau}$.

A. Time Delay Estimation

The following lemma is instrumental to deriving the ML estimate of the delay vector.

Lemma 1: For any given set of functions $\{g_i(t)\}_{i=1}^N$ where $g_i(t) : \mathbb{R} \to \mathbb{C}$ and $\alpha, t \in \mathbb{R}$, we have

$$\max_{t_2, \ldots, t_N \in \mathbb{R}} \left| \sum_{i=1}^{N} e^{j\alpha t_i} g_i(t_1) \right| = \sum_{i=1}^{N} |g_i(t_1)| \, .$$

The main result of this section is stated in the next theorem.

Theorem 2: The ML estimate of $\boldsymbol{\tau}$ for point targets is given by

$$\hat{\boldsymbol{\tau}} = \arg\max_{\boldsymbol{\tau}} \frac{\left| \sum_{m=1}^{N_t} \sum_{n=1}^{N_r} \frac{e^{j2\pi f_c \tau_{m,n}}}{\tau_{m,n}^{\beta}} \sum_{k=1}^{K} r_n[k] \, s_m^*[k; \tau_{m,n}] \right|^2}{\sum_{m=1}^{N_t} \sum_{n=1}^{N_r} \frac{1}{\tau_{m,n}^{2\beta}}}.$$

B. Target Detection

Based on the ML estimate of $\boldsymbol{\tau}$ provided in Theorem 2 the optimal detector is characterized by the following lemma.

Lemma 2: The optimal test for point targets and for the given estimate $\hat{\boldsymbol{\tau}}$ is

$$\left|\sum_{m=1}^{N_t}\sum_{n=1}^{N_r}\frac{e^{j2\pi f_c\hat{\tau}_{m,n}}}{\hat{\tau}_{m,n}^{\beta}}\sum_{k=1}^{K} r_n[k]\ s_m^*[k;\hat{\tau}_{m,n}]\right|^2 \underset{\mathcal{H}_0}{\overset{\mathcal{H}_1}{\gtrless}} \theta.$$

Proof: As we are estimating $\boldsymbol{\tau}$ jointly with ζ, by setting $\boldsymbol{x} = [\boldsymbol{\tau},\ \zeta]$ and applying Theorem 1 the optimal detector is given by

$$\begin{aligned}\frac{\max_{\boldsymbol{x}}\pi(\boldsymbol{x})f_1(\boldsymbol{R}\mid\boldsymbol{x})}{f_0(\boldsymbol{R})} &= \frac{\max_{\boldsymbol{\tau},\zeta} f_1(\boldsymbol{R}\mid\boldsymbol{\tau},\zeta)}{f_0(\boldsymbol{R})}\\ &= \frac{f_1(\boldsymbol{R}\mid\hat{\boldsymbol{\tau}},\hat{\zeta})}{f_0(\boldsymbol{R})} \underset{\mathcal{H}_0}{\overset{\mathcal{H}_1}{\gtrless}} \lambda,\end{aligned}$$

By using the estimates $\hat{\boldsymbol{\tau}}$ given in Theorem 2 and similarly finding $\hat{\zeta}$, the optimum test for a point target is given by

$$\begin{aligned}&\log\left(\frac{f_1(\boldsymbol{R}\mid\hat{\boldsymbol{\tau}},\hat{\zeta}\ )}{f_0(\boldsymbol{R})}\right) =\\ &\log\left(\frac{e^{-\sum_k\|\boldsymbol{r}[k]\|^2}e^{\frac{E}{T_s\ N_t}|\hat{\zeta}\ |^2\ \sum_{m=1}^{N_t}\sum_{n=1}^{N_r}\ \frac{c^{-2\beta}}{\hat{\tau}_{m,n}^{2\beta}}}}{e^{-\sum_k\|\boldsymbol{r}[k]\|^2}}\right) =\\ &\frac{\left|\sum_{m=1}^{N_t}\sum_{n=1}^{N_r}\ \frac{c^{-\beta}}{\hat{\tau}_{m,n}^{\beta}}\ e^{j2\pi f_c\hat{\tau}_{m,n}}\ \sum_{k=1}^{K} r_n[k]\ s_m^*[k;\hat{\tau}_{m,n}]\right|^2}{\frac{1}{T_s}\sum_{m=1}^{N_t}\sum_{n=1}^{N_r}\ \frac{c^{-2\beta}}{\hat{\tau}_{m,n}^{2\beta}}}\\ &\underset{\mathcal{H}_0}{\overset{\mathcal{H}_1}{\gtrless}} \lambda. \qquad (11)\end{aligned}$$

Moreover, by setting $\theta \triangleq \frac{\sum_{m=1}^{N_t}\sum_{n=1}^{N_r}\ (c\hat{\tau}_{m,n})^{-2\beta}}{T_s}\log\lambda$, the test can be cast as

$$\left|\sum_{m=1}^{N_t}\sum_{n=1}^{N_r}\frac{e^{j2\pi f_c\hat{\tau}_{m,n}}}{\hat{\tau}_{m,n}^{\beta}}\sum_{k=1}^{K} r_n[k]\ s_m^*[k;\hat{\tau}_{m,n}]\right|^2 \underset{\mathcal{H}_0}{\overset{\mathcal{H}_1}{\gtrless}} \theta.$$

In order to determine the value of the threshold θ, note that $e^{j2\pi f_c\hat{\tau}_{m,n}}\sum_{k=1}^{K} r_n[k]\ \ s_m^*[k;\hat{\tau}_{m,n}]$ is distributed as $\mathcal{CN}(0,\frac{1}{T_s})$ under $\mathcal{H}_0$ and $\sum_{k=1}^{K} r_n[k]\ \ s_m^*[k;\hat{\tau}_{m,n}]$ is independent of $\sum_{k=1}^{K} r_{n'}[k]\ \ s_{m'}^*[k;\hat{\tau}_{m',n'}]$ for $m\neq m'$ or $n\neq n'$. As a result, under $\mathcal{H}_0$ we have

$$\begin{aligned}&\sum_{m=1}^{N_t}\sum_{n=1}^{N_r}\frac{e^{j2\pi f_c\hat{\tau}_{m,n}}}{\hat{\tau}_{m,n}^{\beta}}\sum_{k=1}^{K} r_n[k]\ s_m^*[k;\hat{\tau}_{m,n}]\\ &\qquad\sim\mathcal{CN}\left(0,\frac{1}{T_s}\sum_{m=1}^{N_t}\sum_{n=1}^{N_r}\frac{1}{\hat{\tau}_{m,n}^{2\beta}}\right);\end{aligned}$$

and consequently

$$\begin{aligned}&\left|\sum_{m=1}^{N_t}\sum_{n=1}^{N_r}\frac{e^{j2\pi f_c\hat{\tau}_{m,n}}}{\hat{\tau}_{m,n}^{\beta}}\sum_{k=1}^{K} r_n[k]\ s_m^*[k;\hat{\tau}_{m,n}]\right|^2\\ &\qquad\sim\text{Exponential}\left(\frac{T_s}{\sum_{m=1}^{N_t}\sum_{n=1}^{N_r}\frac{1}{\hat{\tau}_{m,n}^{2\beta}}}\right).\end{aligned}$$

Therefore, for a given value of P_{fa}, the threshold level θ is found as

$$\theta = \frac{1}{T_s}\sum_{m=1}^{N_t}\sum_{n=1}^{N_r}\frac{1}{\hat{\tau}_{m,n}^{2\beta}}\log\frac{1}{P_{fa}}.$$

■

IV. Optimal Joint Estimation/Detection for Phased-array Radar

We will compare the performance of MIMO radars against that of conventional phased-array radar systems. A phased-array system utilizes an array of closely-located antennas and therefore, the channel coefficients $h_{m,n}$ are fully correlated, i.e., $h_{m,n} = h$. Moreover, all the emitted waveforms are equal up to a scaling factor, i.e., $s_m(t) = \hat{s}_m s(t)$ for $m = 1,\ldots,N_t$ such that $\sum_{m=1}^{N_t}|\hat{s}_m|^2 = N_t$ (for having the total transmitted energy equal to E). Therefore, by using (5) and the narrow-band assumption, the system model is, for $k = 1,\ldots,K$, given by

$$\begin{aligned}r_n[k] &= \sqrt{\frac{E}{N_t}}h\ s[k;\tau_{1,1}]\sum_{m=1}^{N_t}\frac{c^{-\beta}}{\tau_{m,n}^{\beta}}\hat{s}_m e^{j2\pi f_c(\tau_{1,1}-\tau_{m,n})}\\ &+\ z_n[k] \qquad (12)\end{aligned}$$

For the purpose of analyzing the diversity gains as given in Section V, as well as comparing estimation performance, we provide the optimal detector and estimator for phased-array radars in the following lemma. For further analysis we define $S_n(\boldsymbol{\tau}) \triangleq \sum_{m=1}^{N_t}\frac{1}{(c\tau_{m,n})^{\beta}}\hat{s}_m e^{j2\pi f_c(\tau_{1,1}-\tau_{m,n})}$ and $\hat{S}_n(\boldsymbol{\tau}) \triangleq \sum_{m=1}^{N_t}\frac{1}{(c\tau_{m,n})^{\beta}}\hat{s}_m e^{j2\pi f_c(\tau_{1,1}-2\tau_{m,n})}$.

Lemma 3: The ML estimate of the time-delay vector $\boldsymbol{\tau}$ in phased-array radars for point targets is

$$\hat{\boldsymbol{\tau}} = \arg\max_{\boldsymbol{\tau}}\frac{\left|\sum_{n=1}^{N_r}\hat{S}_n^*(\boldsymbol{\tau})\sum_{k=1}^{K} r_n[k]s^*[k;\tau_{1,1}]\right|^2}{\frac{1}{T_s}\sum_{n=1}^{N_r}|\hat{S}_n(\boldsymbol{\tau})|^2}.$$

Based on the ML estimate of $\boldsymbol{\tau}$ provided in Lemma 3 the optimal detectors are presented in the following lemma.

Lemma 4: The optimal test for the given estimate $\boldsymbol{\tau}$ is given by

$$\left|\sum_{n=1}^{N_r}S_n^*(\boldsymbol{\tau})\sum_{k=1}^{K} r_n[k]s^*[k;\hat{\tau}_{1,1}]\right|^2 \underset{\mathcal{H}_0}{\overset{\mathcal{H}_1}{\gtrless}} \hat{\theta}.$$

V. Diversity Gain Analysis

In the previous sections we have provided closed-form expressions for the time-delay estimators as well as the optimal detectors. In order to quantitatively compare the performance of MIMO and phased-array radars, we analyze how fast their corresponding mis-detection probabilities decay as the transmission SNR increases. The counterpart of this notion in MIMO communication systems is referred to as the *diversity gain*.

In what follows, we say two functions $f_1(x)$ and $f_2(x)$ are *exponentially equal* when $\lim_{x\to\infty}\frac{\log f_1(x)}{\log f_2(x)} = 1$ and denote it by $f_1(x) \doteq f_2(x)$. We will use the following lemma in analyzing the diversity gain of MIMO and phased-array radars.

Lemma 5: For any M independent Gaussian random variables $Y_m \sim \mathcal{N}(\rho\cdot\mu_m, \sigma^2)$, $m = 1,\dots,M$, where $\rho \in \mathbb{R}^+$ and $\mu_m \sim \mathcal{N}(0,\sigma_m^2)$, and for any given $\gamma \in \mathbb{R}^+$, in the asymptote of large values of ρ we have

$$\mathbb{E}_{\boldsymbol{\mu}}\left[\Pr\left(\sum_{m=1}^{M} Y_m^2 < \gamma\right)\right] \doteq \rho^{-M}, \qquad (13)$$

where $\boldsymbol{\mu} \triangleq [\mu_1,\dots,\mu_M]$.

By using the lemma above, in the following theorem we establish the diversity gain achieved by the MIMO and phased-array radars for point targets. We denote the probability of mis-detecting a target at the signal-to-noise ratio SNR by $P_{md}(\mathsf{SNR})$.

Theorem 3: The diversity gain achieved by

1) an $N_t \times N_r$ MIMO radar system for point targets is 1, i.e., $P_{md}^{\rm E}(\mathsf{SNR}) \doteq \mathsf{SNR}^{-1}$;
2) an $N_t \times N_r$ MIMO radar system for point targets is 1, i.e., $P_{md}^{\rm P}(\mathsf{SNR}) \doteq \mathsf{SNR}^{-1}$;

According to Theorem 3, as long as diversity gain for *point* targets is considered, both MIMO radar and phased-array radar are identical.

VI. Simulation Results

We consider different configurations and for the MIMO radar we assume that the transmit and receive antennas are located at $X_m^t = (m,0,0)$ for $m = 1,\dots,4$ and $X_n^r = (0,n,0)$ for $n = 1,\dots,8$, respectively. For the phased-array radar we assume that the transmit antennas are all closely-located around $X_m^t = (1,0,0)$ for $m = 1,\dots,4$, and the receive antennas are closely-located around $X_n^r = (0,1,0)$ for $n = 1,\dots,8$. Also we assume that the target to be detected is located at $X_0 = (20,15,0)$, where all the distances are in kilometer (km). The path loss coefficient is $\beta = 2$ and the carrier frequency is $f_c = 5$ MHz. We assume that the target comprises of $P = 10$ scatterers and the number of signal samples is $K = 40$.

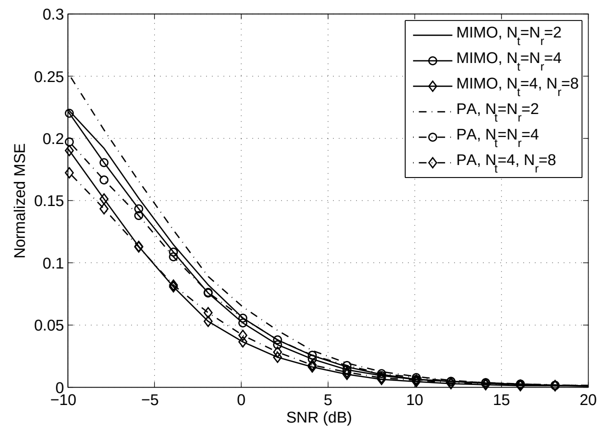

Fig. 1. Average normalized MSE of time-delay estimates versus SNR for point targets.

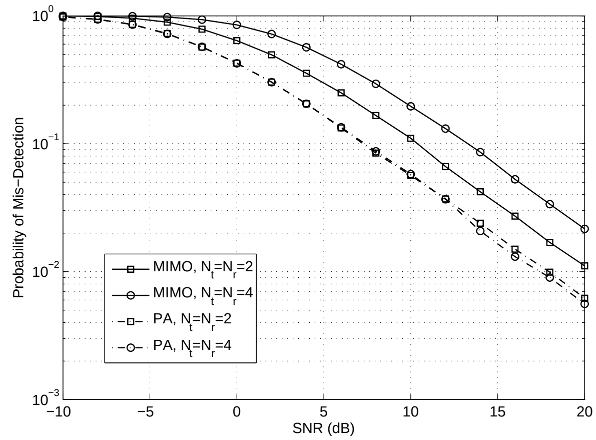

Fig. 2. Probability of mis-detection versus SNR for point targets and $P_{fa} = 10^{-6}$ for point targets.

Finally, for the MIMO radar the emitted waveforms are $s_m(t) = \frac{1}{\sqrt{T}}\exp\left(\frac{j2\pi mt}{T}\right)(U(t) - U(t-T))$, where $U(t)$ is the unit step function and T denotes the duration of the waveform and the sampling rate at the receiver is $T_s = \frac{T}{10}$. For the phased-array radar all the emitted waveforms are equal to $s_1(t)$.

Fig. 1 illustrates the normalized average MSE in estimating the time-delays versus SNR. It is observed that for point targets, conventional phased-array and MIMO radars exhibit similar target detection and time-delay estimation accuracies. Therefore, when considering joint target detection and time-delay estimation, deploying MIMO radars are not much advantageous for point targets. This conclusion is nevertheless limited to the specific problem analyzed in this paper and MIMO radars can be potentially advantageous in other scenarios like those discussed in [4] and references therein. This is due to the fact that point targets lack inde-

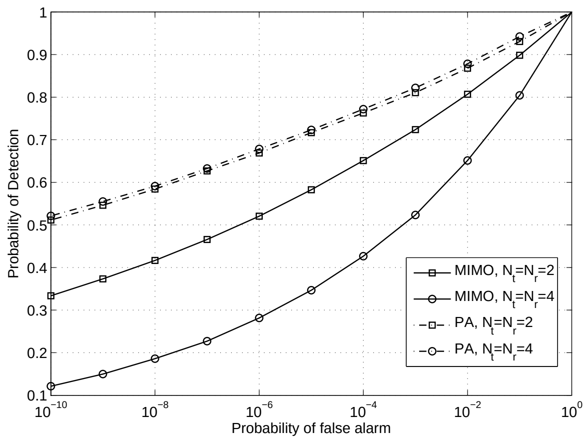

Fig. 3. Probability of target detection versus probability of false alarm for point targets and SNR=0 dB for point targets.

pendent scattering section and thereof provide no diversity gain.

In Fig. 2 and Fig. 3 the probability of mis-detection versus SNR and the ROC are plotted, respectively. For Fig. 2 the tests are designed such that the probability of false alarm is $P_{fa} \leq 10^{-6}$ and for Fig. 3 we have set SNR=0 dB. It is seen from Fig. 2 that both the phase-array and the MIMO radars exhibit a diversity gain of 1, which verifies Theorem 3.

VII. Conclusions

In this paper we have first treated the problem of detecting a *point* target while some of its parameters are unknown and have proposed a framework for optimally detecting the target and estimating such parameters. As an example, we have formulated the optimal detectors and estimators for the problem of jointly detecting the target and estimating the time-delays that a transmitted waveform experiences from being emitted by the transmit antennas until being received by the receive antennas. Secondly, the analysis of the diversity gain, which we have defined as the rate that the probability of mis-detecting the target decays with increasing SNR.

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