Finite-sample Optimal Joint Target Detection and Parameter Estimation by MIMO Radars

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***Abstract*—We consider MIMO radar systems with widely-spaced antennas. We treat the problem of detecting extended targets when one or more target parameters of interest are unknown. We provide a composite hypothesis testing framework for jointly detecting the target along with such parameter estimation while only a *finite* number of signal samples are available. The test offered is *optimal* in a Neyman-Pearson-*like* sense such that it offers a Bayesian-optimal detection test, minimizes the average maximum likelihood parameter estimation error subject to an upper bound constraint on the false-alarm probability, and requires a finite number of samples. While the test can be applied for concurrently detecting the target along with estimating any unknown parameter of interest, we consider the problem of detecting a target which lies in an unknown space range and find the range through estimating the time delays that the emitted waveforms undergo from being illuminated to the target until being observed by the receive antennas. We also analyze the *diversity gain* which we define as the rate that the probability of mis-detecting a target decays with the increasing SNR for a controlled false-alarm and show that for a MIMO radar with N_t and N_r transmit and receive antennas, respectively, the diversity gain is $N_t \times N_r$.**

***Index Terms*—Delay estimation, diversity gain, finite samples, GLRT, MIMO radar, target detection.**

I. Introduction

In this paper we consider a widely-spaced antenna configuration and treat two problems. First, we analyze target detection when some radar parameters are unknown and we are interested in estimating them. We offer a framework for joint target detection and parameter estimation in a Neyman-Pearson-*like* sense (exact definition of optimality is provided in Section II-B) when the receive antennas can acquire only a *finite* number of observations. While the proposed framework can be exploited for detecting the target in conjunction with estimating any parameter of interest, we consider estimating the time-delays that the transmitted waveforms undergo before reaching the receive antennas. While such estimation is necessary optimal detection, they also can be manipulated in order to estimate the range of the target. We formulate this problem as a composite hypothesis test which is shown to be solved optimally via the widely-known generalized likelihood ratio test (GLRT). Note that the existing optimality results of the GLRT hold for the asymptote of an *infinite* number of observations under certain assumptions [1, Sec. 5.6].

As the second problem, we adopt the notion of diversity gain from MIMO communications and define it as the rate that the probability of target mis-detection decays with the increasing signal-to-noise ratio (SNR), while the false-alarm probability is kept below a certain level. We analytically assess the diversity gain yielded by MIMO radars as a function of the number of transmit and receive antennas. We treat the same problem for phased-array radars as well to furnish a benchmark for comparisons. The summary of the contributions and results of this work is as follows.

We propose an optimality measure which is shaped by target detection performance, parameter estimation accuracy, and false-alarm probability. The corresponding optimal composite hypothesis test which satisfies all the optimality criteria is introduced and is deployed for detecting a target in an unknown range. The range of the target is estimated via estimating time-delays. The existing literature on MIMO radars with widely-space antennas either embed the effect of the time-delays only as phase-shifts while ignoring the signal attenuation due to path-loss which is also a function of the time delay [2]–[4], or consider identical path-losses corresponding to different time-delays [5] and absorb it in the scatterer's reflectivity factor. We provide a model which considers different path-losses corresponding to different time-delays.

II. System Descriptions

A. MIMO Radar

We consider a MIMO radar system comprising of N_t and N_r transmit and receive antennas, respectively, and adopt the classical Swerling case I model [6] extended for multiple-antenna systems [3], [4]. According to this model, a target consists of one or more small scatterers exhibiting random, independent and isotropic scintillation.

We define P as the number of the target's scatterers and denote their locations in the Cartesian coordination by $X_p \triangleq (x_p, y_p, z_p)$ for $p = 1, \dots, P$. Also, we denote the reflectivity factor of the p^{th} scatterer by ζ_p and assume that $\{\zeta_p\}_{p=1}^P$ are identically and independently distributed as zero-mean complex random variables with variance $\frac{1}{P}$, i.e., $\mathbb{E}[|\zeta_p|^2] = \frac{1}{P}$ for $p = 1, \dots, P$. The target and reflectivity factors are assumed to remain constant during a finite number of observations denoted by K and change to independent states afterwards. Motivated by capturing the inherent diversity provided by independent scatterers, the antennas are spaced widely enough (such that they satisfy the conditions in [3, Sec. II.A]) to ensure having uncorrelated reflections from the target to the receive antennas. We assume that the transmit antennas are located at $X_m^t \triangleq (x_m^t, y_m^t, z_m^t)$, for $m = 1, \ldots, N_t$, and the receive antennas are located at $X_n^r \triangleq (x_n^r, y_n^r, z_n^r)$, for $n = 1, \ldots, N_r$. The transmit antennas emit N_t *narrowband* waveforms of duration T whose lowpass equivalents are given by $\sqrt{\frac{E}{N_t}} s_m(t)$ for $m = 1, \ldots, N_t$, where E is the total transmitted energy and $\int_T |s_m(t)|^2 dt = 1$. In contrast to the conventional phased-array radars which deploy waveforms which are identical upto a scaling factor [7], in MIMO radar systems these waveforms are designed such that they facilitate acquiring independent observations of each scatterer and often are assumed to be orthonormal, i.e.,

$$\int_T s_m(t) s_n^*(t)\, dt = \delta(m-n), \tag{1}$$

where * denotes complex conjugate and $\delta(\cdot)$ is the Dirac's delta function. The waveform illuminated by the m^{th} transmit antenna to the p^{th} scatterer and received by the n^{th} receive antenna passes through an end-to-end distance which we denote by $d_{m,n}^p$ and undergoes a time delay which we denote by $\tau_{m,n}^p \triangleq d_{m,n}^p / c$, where c is the speed of light. By defining β as the path-loss exponent and superimposing the effects of all scatterers, the base-band equivalent of the signal received by the n^{th} receive antenna due to the waveform $s_m(t)$ transmitted by the m^{th} transmit antenna is given by

$$\begin{aligned} r_{m,n}(t) \;&=\; \sqrt{\frac{E}{N_t}} \sum_{p=1}^{P} \zeta_p \Big(\frac{1}{d_{m,n}^p}\Big)^{\beta} s_m(t-\tau_{m,n}^p) e^{-j2\pi f_c \tau_{m,n}^p} \\ &+\; z_{m,n}(t). \end{aligned} \tag{2}$$

Note that this model differs from those of [3] and [4] in the sense that we have added the attenuation effects of path-losses by including the terms $\Big(\frac{1}{d_{m,n}^p}\Big)^{\beta} = \Big(\tau_{m,n}^p\ c\Big)^{-\beta}$. The exponential term $\exp(-j2\pi f_c \tau_{m,n}^p)$ in (2) represents the effect of propagation phase shift, where f_c is the carrier frequency, and $z_{m,n}(t) \sim \mathcal{CN}(0, \frac{1}{N_t})$, denotes the additive white Gaussian noise.

We define $X_0 \triangleq (x_0, y_0, z_0)$ as the location of the gravity center of the target and denote its associated time delays and distances by $\tau_{m,n}$ and $d_{m,n}$, respectively. We assume that the distances $d_{m,n}$ are considerably larger than the dimensions of the object such that we can replace the distances and the time-delays associated with the scatterer X_p with those corresponding to the gravity center of the target X_0, i.e., $d_{m,n}^p = d_{m,n}$ and $\tau_{m,n}^p = \tau_{m,n}, \forall p$. Therefore, for $m = 1, \ldots, N_t$, $n = 1, \ldots, N_r$ and $p = 1, \ldots, P$,

$$s_m(t-\tau_{m,n}^p) = s_m(t-\tau_{m,n}). \tag{3}$$

Using (2)-(3), the received signal at the n^{th} antenna, which is a superposition of all emitted waveforms, is given by

$$r_n(t) \;=\; \sqrt{\frac{E}{N_t}} \sum_{m=1}^{N_t} \frac{c^{-\beta}}{\tau_{m,n}^{\beta}}\, s_m(t-\tau_{m,n}) h_{m,n} + z_n(t),$$

where $h_{m,n} \triangleq \sum_{p=1}^{P} \zeta_p\ e^{-j2\pi f_c \tau_{m,n}^p}$ and $z_n(t) \triangleq \sum_{m=1}^{N_t} z_{m,n}(t) \sim \mathcal{CN}(0,1)$. Furthermore, we assume that the waveforms are narrowband. Based on the narrow-band assumption, for $m = 1, \ldots, N_t$ and $n = 1, \ldots, N_r$ [4], we get

$$\forall \tau, \quad s_m(t) = e^{j2\pi f_c \tau}\ s_m(t-\tau), \tag{4}$$

which in conjunction with the orthonormality assumption (1) implies that

$$\forall\ \tau_{m,k}, \tau_{n,k},\ \int_T s_m(t-\tau_{m,k}) s_n^*(t-\tau_{n,k})\, dt = \delta(m-n).$$

We also define the time-delay vector $\boldsymbol{\tau} \triangleq [\tau_{1,1}, \ldots, \tau_{N_t,N_r}]$. Based on the model given in (2) and noting that the noise-terms are unit-variance, the transmission signal-to-noise ratio, denoted by SNR, is given by $\mathsf{SNR} = \frac{E}{T}$

B. Problem Statement

We assume that the receive antennas sample the received signal at the rate of $\frac{1}{T_s}$ samples per second. By defining $r_n[k] \triangleq r_n(kT_s)$, $z_n[k] \triangleq z_n(kT_s)$ and $s_m[k;\tau] \triangleq s_m(kT_s - \tau)$, the discrete-time low-pass equivalent of the received signal when a target is present is, for $k = 1, \ldots, K$, given by

$$r_n[k] = \sqrt{\frac{E}{N_t}}\ c^{-\beta} \sum_{m=1}^{N_t} \frac{1}{\tau_{m,n}^{\beta}}\ h_{m,n} s_m[k;\tau_{m,n}] + z_n[k]. \tag{5}$$

We also assume that the sampling rate is high enough to ensure that the discrete-time signals $s_m[k;\tau_{m,n}]$ remain orthogonal for arbitrary delays $\tau_{m,l}, \tau_{n,l}$, i.e., $\sum_k s_m[k;\tau_{m,l}] s_n^*[k;\tau_{n,l}] = \frac{1}{T_s}\delta(m-n)$. Let us define $\boldsymbol{r}[k] \triangleq [r_1[k], \ldots, r_{N_r}[k]]^T$ for $k = 1, \ldots, K$ and $\boldsymbol{R} \triangleq [\boldsymbol{r}[1]^T, \ldots, \boldsymbol{r}[K]^T]^T$. Also let $f_0(\boldsymbol{R})$ denote the probability density function (pdf) of the received signal when a target is not present. When a target is present, the pdf of the received signal depends on the unknown parameter $\boldsymbol{\tau}$ and is denoted by $f_1(\boldsymbol{R} \mid \boldsymbol{\tau})$. Therefore, by defining the estimate of $\boldsymbol{\tau}$ by $\widehat{\boldsymbol{\tau}}$, the detection part of the problem can be cast as

$$\begin{cases} \mathcal{H}_0 : \text{No target exists at delay } \widehat{\boldsymbol{\tau}} \text{ where} \quad \boldsymbol{R} \sim f_0(\boldsymbol{R}), \\ \mathcal{H}_1 : \text{Target exists at delay } \widehat{\boldsymbol{\tau}} \text{ where} \quad \boldsymbol{R} \sim f_1(\boldsymbol{R} \mid \widehat{\boldsymbol{\tau}}). \end{cases} \tag{6}$$

Our objective is to detect a target when the vector time-delays $\boldsymbol{\tau}$ is unknown such that that the following conditions are satisfied.

- C1) The average ML estimation error of the time delays $\boldsymbol{\tau}$ is minimized.
- C2) The false-alarm probability of the target detector is kept below a certain level.
- C3) For the given set of ML estimates $\widehat{\boldsymbol{\tau}}$, the target detector is Bayesian-optimal, i.e., the Bayes risk is minimized [8, Sec. II.B].
- C4) The test requires only a finite number of samples, i.e., $K < \infty$.

We call the composite hypothesis test that satisfies all the conditions above Neyman-Pearson-*like* optimal as it follows the same spirit as the standard Neyman-Pearson (NP) criterion (minimize the detection error probability subject to a constraint on false-alarm probability).

Compared to the existing literature, e.g., [3] and references therein, in addition to the objectives (conditions C1-C4) and the model (including path-loss effects) our proposed framework has also a slightly different application. The existing models aim at detecting a target given that it lies within a given range, while in this paper we aim at estimating the range for the potentially existing target within any arbitrary subspace of the entire search space and detecting the presence of the target based on those estimates.

Note that when we are only interested in hypothesis testing (and not in estimating $\boldsymbol{\tau}$), the optimal test is given by

$$\frac{f_1(\boldsymbol{R})}{f_0(\boldsymbol{R})} = \frac{\int \pi(\boldsymbol{\tau}) f_1(\boldsymbol{R} \mid \boldsymbol{\tau}) d\boldsymbol{\tau}}{f_0(\boldsymbol{R})} \underset{\mathcal{H}_0}{\overset{\mathcal{H}_1}{\gtrless}} \lambda, \tag{7}$$

where λ is found by the conditions enforced on the tolerable level of false alarm. On the other hand, the optimal test that satisfies all the conditions C1-C4 given above is characterized by the following Theorem.

Theorem 1 (Moustakides [9]): For a finite cardinality vector $\boldsymbol{R} \in \mathbb{C}^{|\boldsymbol{R}|}$ and an unknown vector parameter $\boldsymbol{x}$, the optimal test in the NP-*like* sense (that satisfies C1-C4) for estimating $\boldsymbol{x}$ and deciding between $\mathcal{H}_0$ and $\mathcal{H}_1$ given as

$$\begin{cases} \mathcal{H}_0: & \boldsymbol{R} \sim f_0(\boldsymbol{R}), \\ \mathcal{H}_1: & \boldsymbol{R} \sim f_1(\boldsymbol{R} \mid \boldsymbol{x}), \end{cases} \tag{8}$$

is

$$\frac{f_1(\boldsymbol{R} \mid \widehat{\boldsymbol{x}})}{f_0(\boldsymbol{R})} \underset{\mathcal{H}_0}{\overset{\mathcal{H}_1}{\gtrless}} \lambda, \quad \text{where} \quad \widehat{\boldsymbol{x}} = \arg\max_{\boldsymbol{x}} \pi(\boldsymbol{x}) f_1(\boldsymbol{R} \mid \boldsymbol{x}), \tag{9}$$

and $\pi(\boldsymbol{x})$ is the prior distribution of $\boldsymbol{x}$.

In the theorem above, $\widehat{\boldsymbol{x}}$ given in (9) is the *maximum a posteriori* (MAP) estimate of $\boldsymbol{x}$. As we do not have any prior information about the location of the target, throughout the analysis we assume that $\pi(\boldsymbol{\tau})$ has a uniform distribution. Hence, the MAP estimate of $\boldsymbol{\tau}$ becomes its ML estimate. The above Theorem essentially establishes the GLRT as the optimal estimation/detection strategy that satisfies the conditions C1-C4. Several other *asymptotic* optimality results are known for the GLRT which are all based on having an *infinite* number of observation [9] and [10, Chapter 22].

We will analyze this detection/estimation problem for extended targets which consist of many scatterers, i.e., $P \gg 1$. This corresponds to targets with large dimensions, like land or ocean surfaces, that cannot be modeled with a single scatterer and therefore are modeled as a group of scatterers.

III. Joint Detection and Estimation for Extended Targets

In this section we consider targets which are extended enough to be modeled as a group of isotropic and independent scatterers, i.e., $P \gg 1$ in (2). Our objective is to optimally (in the NP-like sense) detect a target, based on the model in (6), and simultaneously estimate the vector of time delays $\boldsymbol{\tau}$ with a finite number of observations. According to Theorem 1 this can be solved via the GLRT given in (9) for $\boldsymbol{x} = \boldsymbol{\tau}$.

We start by deriving the ML estimate of $\boldsymbol{\tau}$ and then provide the detection-related analysis.

A. Time Delay Estimation

The hypothesis test is

$$\begin{cases} \mathcal{H}_0: r_n[k] = z_n[k], \\ \mathcal{H}_1: r_n[k] = \sqrt{\frac{E}{N_t}} \sum_{m=1}^{N_t} \frac{c^{-\beta}}{\tau_{m,n}^{\beta}} h_{m,n} s_m[k;\tau_{m,n}] + z_n[k]. \end{cases} \tag{10}$$

Denote $\boldsymbol{h} \triangleq [h_{1,1}, h_{1,2}, \ldots, h_{N_t,N_r}]^T$, where $h_{m,n}$, as defined in (4), accounts for the effects of the position and reflectiveness of the scatterers corresponding to the m^{th} transmit and the n^{th} receive antenna. The antennas are widely separated and the reflectivity factors $\{\zeta_p\}$ are complex and independently distributed with zero mean and variance $\frac{1}{P}$ and $P \gg 1$. Therefore, by also using the central limit theorem, $\{h_{m,n}\}$ are i.i.d. and distributed as $\mathcal{CN}(0,1)$ [3].

Based on the model given in (5), the vector of the received signals $\boldsymbol{R}$ depends on the time-delays, which we are interested in estimating, as well as the unknown random vector $\boldsymbol{h}$ and its pdf for any given $\boldsymbol{\tau}$ and $\boldsymbol{h}$ is $f_1(\boldsymbol{R} \mid \boldsymbol{\tau}, \boldsymbol{h})$. In order to obtain the ML estimate of $\boldsymbol{\tau}$ we have to either estimate it through solving $\arg\max_{\boldsymbol{\tau}} f_1(\boldsymbol{R} \mid \boldsymbol{\tau})$ which requires recovering $f_1(\boldsymbol{R} \mid \boldsymbol{\tau})$ from $f_1(\boldsymbol{R} \mid \boldsymbol{\tau}, \boldsymbol{h})$ by averaging over all realizations of $\boldsymbol{h}$ or jointly estimate it with $\boldsymbol{h}$ when deemed to be beneficial. Estimating $\boldsymbol{h}$ is more beneficial than averaging over all realizations of $\boldsymbol{h}$ when an accurate estimate of $\boldsymbol{h}$ is available, e.g., in high SNR regimes, while averaging leads to a better performance when the estimate is very inaccurate, e.g., in low SNR regimes. The ML estimate of $\boldsymbol{\tau}$ is provided in the following theorem for both scenarios.

Theorem 2: The ML estimate of $\tau_{m,n}$ for extended targets

1) through MAP estimation of $\boldsymbol{h}$ is given by

$$\forall\, m,n \ \widehat{\tau}_{m,n}^{\mathrm{MAP}} = \arg\max_{\tau_{m,n}} \left\{ \frac{\left| \sum_{k=1}^{K} r_n[k]\ s_m^*[k;\tau_{m,n}] \right|^2}{\frac{1}{T_s} + \frac{N_t}{E} (c\tau_{m,n})^{2\beta}} \right\},$$

2) and through averaging over all realizations of $\boldsymbol{h}$ is given by

$$\forall\, m,n \ \widehat{\tau}_{m,n}^{\mathrm{ave}} = \arg\max_{\tau_{m,n}} \left\{ \frac{\left| \sum_{k=1}^{K} r_n[k]\ s_m^*[k;\tau_{m,n}] \right|^2}{\frac{1}{T_s} + \frac{N_t}{E} (c\tau_{m,n})^{2\beta}} - \frac{1}{2} \log\left(\frac{E}{T_s N_t} (c\tau_{m,n})^{-2\beta} + 1 \right) \right\}.$$

Proof: See [11]. ■

According to the theorem above, estimating the vector of time-delays $\boldsymbol{\tau}$ boils down to estimating individual time-delays $\tau_{m,n}$ independently. Decoupled estimation translates into less computational cost, especially when $(N_t N_r)$ is large. Estimating individual time-delays can be implemented via a correlator and a multiplier for computing $\sum_{k=1}^{K} r_n[k]\ s_m^*[k;\tau_{m,n}]$ and incorporating the effect of $\tau_{m,n}$ in the denominator, respectively, followed by a search for all values of $\tau_{m,n}$.

B. Target Detection

Based on the ML estimates of the delay vector $\boldsymbol{\tau}$ provided in Theorem 2, we proceed to find their corresponding optimum detectors. We show that both estimates give rise to the same optimal detector given in the following lemma.

Lemma 1: The optimal test for extended targets and for the given estimate $\widehat{\boldsymbol{\tau}}$ is

$$\sum_{m=1}^{N_t}\sum_{n=1}^{N_r}\frac{\left|\sum_{k=1}^{K} r_n^*[k]\ s_m[k;\widehat{\tau}_{m,n}]\right|^2}{\frac{E}{T_s N_t}+(c\widehat{\tau}_{m,n})^{2\beta}} \underset{\mathcal{H}_0}{\overset{\mathcal{H}_1}{\gtrless}} \theta. \tag{11}$$

Proof: See [11]. ■

It is noteworthy that our detection scheme has two major differences with that of [3] provided in [3, Eq. (24)]. First, the detection formulation in [3, Eq. (24)] tests whether a target exists at a *known* delay while we try to detect a target with *unknown* delays. Secondly, the model of [3] embeds the effect of the time-delays only as *phase shifts* and ignores the *path-loss* effect. By recalling that the path losses also depend on the time delays we have modified the model of [3] to also capture the effects of path losses in our model. Note that the path losses are of the form $(c\tau_{m,n})^{-\beta}$ and if we eliminate the path loss effect (as done in [3]) by setting $\beta = 0$ for instance, our detector provided in (11) becomes exactly equivalent to that of [3].

IV. Diversity Gain Analysis

In the previous sections we have provided closed-form expressions for the time-delay estimators as well as the optimal detectors. In order to quantitatively compare the performance of MIMO and phased-array radars, we analyze how fast their corresponding mis-detection probabilities decay as the transmission SNR increases. The counterpart of this notion in MIMO communication systems is referred to as the *diversity gain*.

In what follows, we say two functions $f_1(x)$ and $f_2(x)$ are *exponentially equal* when $\lim_{x\to\infty}\frac{\log f_1(x)}{\log f_2(x)} = 1$ and denote it by $f_1(x) \doteq f_2(x)$.

In the following theorem we establish the diversity gain achieved by the MIMO and phased-array radars for extended targets. We denote the probability of mis-detecting a target at the signal-to-noise ratio SNR by $P_{md}(\mathsf{SNR})$.

Theorem 3: The diversity gain achieved by

1) an $N_t \times N_r$ MIMO radar system for extended targets is $N_t N_r$, i.e., $P_{md}^{\rm E}(\mathsf{SNR}) \doteq \mathsf{SNR}^{-N_t N_r}$;
2) an $N_t \times N_r$ phased array system for extended targets is 1, i.e., $P_{md}^{\rm PA}(\mathsf{SNR}) \doteq \mathsf{SNR}^{-1}$.

Proof: See [11]. ■

According to Theorem 3, while adding transmit or receive antennas in MIMO radar systems leads to more diversity gain and thereof, more reliable target detection, doing so in conventional phased-array systems yields no additional diversity gain. This is intuitively justified by noting that in phased-array radars, as the transmit antennas are located closely they illuminate the target scatterers with essentially identical angles and the receiving antennas receive the reflected waveforms with identical angles. Therefore, from the viewpoint of the scatterers in a phased-array radar, emitting the waveforms via different transmit antennas or summing them up and using only one transmit antenna provide the same diversity gain. The same argument holds for the receive antennas as well.

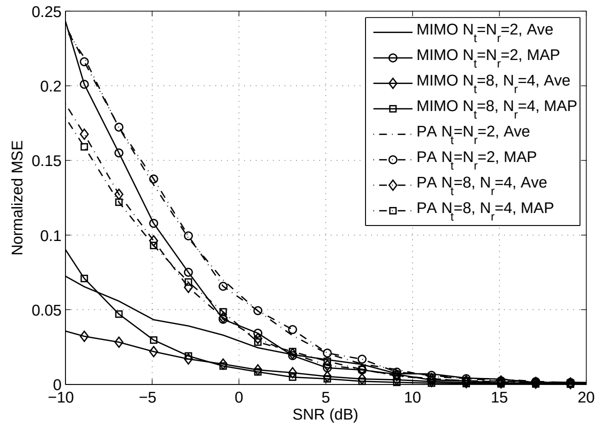

Fig. 1. Average normalized MSE of time-delay estimates versus SNR.

V. Simulation Results

In this section we provide simulation results on the performance of the proposed algorithms in terms of parameter estimation and detection. We consider two antenna configurations with $N_t = N_r = 2$ and $N_t = 4, N_r = 8$ respectively. We assume that the transmit and receive antennas are located at $X_m^t = (m, 0, 0)$ for $m = 1, \ldots, 4$ and $X_n^r = (0, n, 0)$ for $n = 1, \ldots, 8$, respectively, and the target to be detected is located at $X_0 = (20, 15, 0)$ where all the distances are in kilometer (km). The path loss coefficient is $\beta = 2$, the carrier frequency is $f_c = 5$ MHz and we assume that the target comprises of $P = 10$ scatterers and the number of signal samples is $K = 40$. Finally, as the waveform design is beyond the scope of this work, we have set the emitted waveforms to be $s_m(t) = \frac{1}{\sqrt{T}}\exp\left(\frac{j2\pi mt}{T}\right)(U(t) - U(t-T))$, where $U(t)$ is the unit step function and T denotes the duration of the waveform and $T = 10\,T_s$. In the phase-array radar, only $s_1(t)$ is used.

We first consider the performance of parameter estimation. Figure 1 depicts the average normalized mean-square errors (MSE) for phased-array and for MIMO radar, i.e., $\frac{1}{N_t N_r}\sum_m\sum_n\left|\frac{\widehat{\tau}_{m,n}-\tau_{m,n}}{\tau_{m,n}}\right|^2$ as a function of received SNR. It is seen that the MIMO radar outperforms the phased-array in all SNR range and in particular, by a large margin in the low SNR regime, which is of more interest in radar applications. Moreover, it is seen that in MIMO radars, the MAP estimator $\widehat{\boldsymbol{\tau}}^{\rm MAP}$ performs better than the estimator $\widehat{\boldsymbol{\tau}}^{\rm ave}$in the high

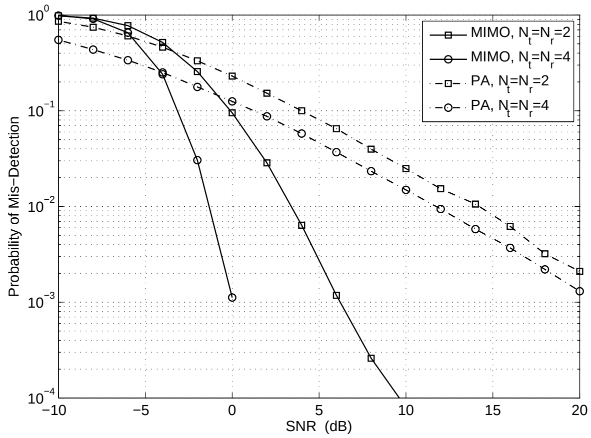

Fig. 2. Probability of mis-detection versus SNR for a false alarm $P_{fa} = 10^{-6}$.

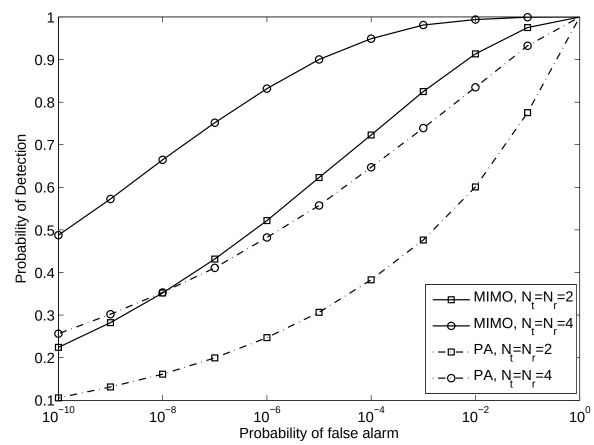

Fig. 3. Probability of target detection versus probability of false alarm for $\mathsf{SNR} = 0$ dB.

SNR regime, while the estimator $\widehat{\boldsymbol{\tau}}^{\text{ave}}$ outperforms the MAP estimator $\widehat{\boldsymbol{\tau}}^{\text{MAP}}$ in the low SNR regime, as expected.

It is seen that the 4×8 MIMO radar performs considerably better than the 2×2 MIMO radar, which is due to the fact that 32 time-delays provide much more information about the position of the target than 4 time-delays do.

In Fig. 2 the probability of mis-detection versus SNR is illustrated. The tests are designed such that the probability of false alarm is $P_{fa} \leq 10^{-6}$. As analyzed in Section IV and observed in this figure, the slope of the mis-detection probability of the phased-array radar is 1 decade per 10 dB, whereas that of the MIMO radar is $N_t N_r$ times steeper. Figure 3 shows the receiver operating curve (ROC) for the MIMO and the phased-array systems, for $\mathsf{SNR} = 0$ dB. It is seen that the MIMO radar significantly outperforms the phased-array radar over a wide range of false alarm values.

VI. Conclusions

In this paper we have first treated the problem of detecting a target while some of its parameters are unknown and have proposed a framework for optimally detecting the target and estimating such parameters. As an example, we have formulated the optimal detectors and estimators for the problem of jointly detecting the target and estimating the time-delays that a transmitted waveform experiences from being emitted by the transmit antennas until being received by the receive antennas. Secondly, the analysis of the diversity gain, which we have defined as the rate that the probability of mis-detecting the target decays with increasing SNR where we have shown that in a $N_t \times N_r$ widely-spaced antenna configuration, the achievable diversity gain is $N_t N_r$.

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