AN L_2 BASED METHOD FOR THE DESIGN OF ONE DIMENSIONAL FIR DIGITAL FILTERS

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ABSTRACT

FIR filters obtained with the classical L_2 method have performance that is very sensitive to the form of the ideal response selected for the transition region. In this paper we propose a means for selecting the unknown part of a complex ideal response optimally. By selecting a proper L_2 criterion and using variational techniques we succeed in minimizing the criterion with respect to the ideal response and thus obtain its corresponding optimum form. The complete solution to the problem can be obtained by solving a simple system of linear equations suggesting a reduced complexity for the proposed method. Using the optimum form of the ideal response we also propose a new suboptimal method for the design of weighted FIR filters. Design examples are presented to illustrate the performance of the proposed method.

1. INTRODUCTION

A very important class of 1-D filters is the class of linear phase Finite Impulse Response (FIR) filters. This class is tractable because the linear phase restriction converts the filter design problem into a real approximation problem. However, the linear phase restriction is not needed in the stopbands of the filter. Imposing the linear phase requirement only inside in the passbands of the filter improves significantly the approximation error. On the other hand the design problem becomes a complex approximation problem. Complex approximation is also needed for the design of filters with nonlinear phase such as FIR equalizers, beamformerms and seismic migration filters.

The most common techniques used for the design of complex FIR filters use as approximation criterion the minimization of the L_2 or L_∞ measure. The L_∞ criterion is considerable more difficult to use in the complex case than it is in the real one. This is because in the complex case the alternation property of the error function is not necessary for optimality [7]. Thus the minimization of the L_∞ measure needs the use of sophisticated optimization tools as iterative constrained linear programming [1], [2], or iterated reweighted least squares [4], that require a large computational effort.

The L_2 criterion is the simplest criterion and results in an easily computable Fourier series approximation. Unfortunately this method is known for its poor performance that is more pronounced at the discontinuity points of the ideal response (Gibb's phenomenon) [9], [10]. The performance of the L_2 method can be improved if transition regions are introduced between passbands and stopbands. There are two categories of design techniques based on this idea. The first includes methods that define the ideal response inside the transition region using some arbitrary class of functions and the second category considers the transition region as "don't care" and simply removes them from the error measure [8], [10].

In [11] a new L_2 method for the design of the zero phase FIR filters was presented. Specificaly, by minimizing a properly selected L_2 measure with respect to the filter coefficients and with respect to the unknown ideal response (using variational techniques [6]) optimum L_2 filters were designed that had a very good performance as compared to the L_∞ measure.

In this paper we extend this idea to the complex FIR filters. In the next section we are going to define our optimization criterion and present the solution.

2. OPTIMIZATION CRITERION AND OPTIMUM APPROXIMATION

Let us consider a complex function $D(\omega)$ that we like to approximate in the L_2 sense using linear combinations of the complex exponentials $e^{jn\omega}$, $n = N_1, \ldots, N_2$ where we assume that $N_2 - N_1$ is an even number (odd length filter). We can easily prove that this problem is the same as approximating $D(\omega)e^{-jT\omega}$ with linear combinations of the exponentials $e^{jn\omega}$, $n = -N, \ldots, N$ where $N = (N_2 - N_1)/2$ and $T = (N_1 + N_2)/2$. So for now on we will assume that we have this case. If we define the vector function

$$\phi(\omega) = [\phi_{-N}(\omega) \ \cdots \ \phi_0(\omega) \ \cdots \ \phi_N(\omega)]^t \qquad (1)$$

with $\phi_n(\omega) = e^{jn\omega}$ then, it is well known that the Fourier approximation of function $D(\omega)$ can be expressed as $H_D(\omega) = \phi^t(\omega)\mathbf{h}_D$ and the optimal coefficient are given by $\mathbf{h}_D =< \phi, D >$ where $<,>$ denotes the usual inner product of two complex functions.

Let us now proceed to the definition of the optimality criterion. To this end let $-\pi = \omega_0 < \omega_1 < \omega_2 < \omega_3 < \ldots < \omega_{M-2} < \omega_{M-1} = \pi$ be any M distinct points on the interval $\mathcal{I} = [-\pi \ \pi]$ and let

$D(\omega)$ be a complex function defined on this interval as follows

$$D(\omega) = \begin{cases} F(\omega) & \omega \in \mathcal{U}_i, \ i = 1, \ldots, N_u \\ G(\omega) & \omega \in \mathcal{T}_i, \ i = 1, \ldots, N_t \end{cases} \quad (2)$$

where $N_u = \lceil \frac{M}{2} \rceil$, $N_t = \lfloor \frac{M}{2} \rfloor$, $\mathcal{U}_i = [\omega_{2(i-1)} \ \omega_{2i-1}]$, $\mathcal{T}_i = (\omega_{2i-1} \ \omega_{2i})$ and $F(\omega)$ is assumed known while $G(\omega)$ is unknown. Let us denote with $\mathcal{U} = \cup_{i=1}^{N_u} \mathcal{U}_i$, $\mathcal{T} = \cup_{i=1}^{N_t} \mathcal{T}_i$. Notice that the region $\mathcal{U}$ is the union of the N_u closed disjoint intervals $\mathcal{U}_i$ where $D(\omega)$ is assumed known, while the region $\mathcal{T}$ is the union of the N_t open disjoint intervals $\mathcal{T}_i$ where $D(\omega)$ is assumed unknown.

Since the part of the complex function $D(\omega)$ inside the region $\mathcal{T}$ is not explicitly given this means that, by varying $G(\omega)$, we can have a whole class of possible functions $D(\omega)$.

As in the real case [11] let us define the following L_2 criterion

$$\mathcal{E}(D, \mathbf{h}) = < D^{(1)} - H_D^{(1)}, D^{(1)} - H_D^{(1)} > \quad (3)$$

where $D^{(1)}$, $H_D^{(1)}$ denote the derivatives of D and H_D respectively. Notice that for a meaningful definition of the criterion $\mathcal{E}(D, \mathbf{h})$ the function $D(\omega)$ must be continuous at all end points ω_i, $i = 1, \ldots, M$.

Our goal now is to minimize $\mathcal{E}(D, \mathbf{h})$ with respect to the coefficients $\mathbf{h}$ of the filter and with respect to the unknown ideal response D. The first minimization with respect to the coefficients (for given D) yields the well known Fourier coefficients. To further minimize the resulting error with respect to D we use similar variational techniques as in [11] and we obtain the following relation for the optimum ideal response $D_o(\omega)$ and its corresponding optimum filter $H_o(\omega)$

$$D_o(\omega) = H_o(\omega) + q_{i,0} + q_{i,1}\omega, \ \omega \in \mathcal{T}_i, \ i = 1, \ldots, N_t \quad (4)$$

We realize from (4) that the optimum form of the complex function inside the region $\mathcal{T}$ is a combination of a regular and a trigonometric polynomial.

To this end, let $\mathbf{h}_o$ denote the Fourier coefficients corresponding to the optimum complex function $D_o(\omega)$, then $H_o(\omega) = \phi^t(\omega)\mathbf{h}_o$. Define now the vector function $\psi(\omega) = [1 \ \omega]^t$ and the N_t vectors $\mathbf{q}_i = [q_{i,0} \ q_{i,1}]^t$. It is then easy to show that we have the following system of equations from which we obtain our unknowns.

$$(2\pi I - A)\mathbf{h}_o - \sum_{i=1}^{N_t} B_i \mathbf{q}_i = \mathbf{h}_{\mathcal{U}} \quad (5)$$

$$\begin{aligned} \phi^t(\omega_{2i-1})\mathbf{h}_o + \psi^t(\omega_{2i-1})\mathbf{q}_i &= F(\omega_{2i-1}-) \\ i &= 1, \ \ldots \ , N_t \end{aligned} \quad (6)$$

$$\begin{aligned} \phi^t(\omega_{2i})\mathbf{h}_o + \psi^t(\omega_{2i})\mathbf{q}_i &= F(\omega_{2i}+) \\ i &= 1, \ldots, N_t \end{aligned} \quad (7)$$

where the involved quantities in the above system are defined as follows

$$\begin{aligned} \mathbf{h}_{\mathcal{U}} &= < \phi, 1_{\mathcal{U}} F > \\ A &= \sum_{i=1}^{N_t} < \phi, 1_{\mathcal{T}_i} \phi^t > \\ B_i &= < \phi, 1_{\mathcal{T}_i} \psi^t > \end{aligned} \quad (8)$$

and $1_{\mathcal{X}}(\omega)$ denotes the index function of the set $\mathcal{X}$. Notice that the $2N+1$ linear equations of (5) result from minimization of the criterion with respect to the coefficients of the filter while the $2N_t$ linear equations of (6) and (7) are obtained by requiring the optimum function D_o to be continuous on the end points of the N_t disjoint intervals $\mathcal{T}_i$. Notice also that all quantities defined in (??) depend only on known functions integrated over known sets and thus can be considered given.

Concluding we obtain the complete solution to the constrained optimization problem by solving the set of linear equations defined by (5), (6) and (7).

2.1. Weighted Least Squares Approximation

There are cases where we are interested in weighting the approximation errors in the bands of interest. This can be taken into account by incorporating a weighting function into the L_2 measure. Generalizing our result of the previous section to the weighted Least Squares (WLS) case was not possible. In other words it was not possible to find a WLS criterion which optimized with respect to the filter coefficients and the unknown ideal response to yield a filter with good performance. On the other hand we are able to propose a method for designing weighted filters that have excellent performance just by properly extending the equations of the previous section to the weighted case. We like to stress that the proposed filter in this section is not optimal in any sense except when the weights are all equal to unity. Thus let us assume that inside the bands of interest we are also given a function $W(\omega)$ which is the necessary weight. The basic idea is to use inside each transition region the following equation

$$\begin{aligned} W(\omega)D_o(\omega) &= W(\omega)H_o(\omega) + q_{i,0} + q_{i,1}\omega \\ & \omega \in \mathcal{T}_i, \ i = 1, \ldots, N_t \end{aligned} \quad (9)$$

corresponding to (4). Notice that in the transition regions $W(\omega)$ is not known. We just define it as a third order polynomial (different in each interval $\mathcal{T}_i$) and such that it insures continuity of $W(\omega)$ and of its derivative. We can show that the resulting $W(\omega)$ is monotone inside each transition region. Following a similar procedure as in the previous section the linear system that gives the solution to our problem is the following

$$(R - A)\mathbf{h}_o - \sum_{i=1}^{N_t} B_i \mathbf{q}_i = \mathbf{h}_{\mathcal{U}} \quad (10)$$

$$\begin{aligned} W(\omega)\phi^t(\omega_{2i-1})\mathbf{h}_o + \psi^t(\omega_{2i-1})\mathbf{q}_i &= v_{2i-1} \\ i = 1, \ \ldots, \ N_t & \end{aligned} \quad (11)$$

$$W(\omega)\phi^t(\omega_{2i})\mathbf{h}_o + \psi^t(\omega_{2i})\mathbf{q}_i = v_{2i} \quad i = 1, \ldots, N_t \tag{12}$$

where

$$\begin{aligned} \mathbf{h}_{\mathcal{U}} &= <\phi, 1_{\mathcal{U}}F>_W \\ R &= <\phi, 1_{\mathcal{I}}\phi^t>_W \\ A &= \sum_{i=1}^{N_t} <\phi, 1_{\mathcal{T}_i}\phi^t>_W \\ B_i &= <\phi, 1_{\mathcal{T}_i}\psi^t>_W \\ v_{2i-1} &= W(\omega_{2i-1}-)F(\omega_{2i-1}-) \\ v_{2i} &= W(\omega_{2i}+)F(\omega_{2i}+) \end{aligned} \tag{13}$$

and $< f,\ g >_W$ denotes the weighted inner product of the complex functions $f,\ g$ defined as $< f,\ g >_W = \int W^2(\omega)f(\omega)g^*(\omega)d\omega$. Notice that all quantities defined in (13) can be computed since they depend only on known functions integrated over known sets.

In the next section we are going to apply our method to the design of complex FIR filters.

3. DESIGN OF FIR DIGITAL FILTERS

Let us assume that the region $\mathcal{U}$ is the union of the passbands and stopbands of the desired filter while the region $\mathcal{T}$ coincides with the union of the transition bands. Under these assumptions it is easy to see that the filter design problem can be considered as a special case of the general approximation problem defined in Section 2.

Let us now apply our method to two different filter design problems and compare it to other existing techniques. Specifically we are going to compare our method against the min-max equiripple [7],[2] and the don't care region [3],[10, p. 70] methods.

Example 1. Let $D_{LP}(\omega)$ be the ideal response of a lowpass filter defined as follows

$$D_{LP}(\omega) = \begin{cases} e^{-j\tau\omega} & |\omega| \in [0\ \omega_p] \\ 0 & |\omega| \in [\omega_s\ \pi] \end{cases} \tag{14}$$

where $\omega_p,\ \omega_s,\ \tau$ are the desired cut-off frequencies and the desired passband group delay of the filter.

Consider the special case $\omega_p = 0.46\ \omega_s = 0.5$ and $\tau = 4N/5$ where ω is normalized in [-1 1]. In Table I we present the maximum ripple e_m in magnitude and the maximum ripple in the passband group delay e_τ for the two methods under comparison, for different values of N. We can conclude from Table I that our method has at least 45% smaller maximum ripple in magnitude as compared to the other method.

Let us now approximate the same ideal response with a filter of length 249 and by using weights 10 and 1 in the stopbands and the passband of the filter respectively. The resulting maximum approximation errors in magnitude and passband group delay are 3.80×10^{-4} and 0.0716 respectively. These results compare favorably with the corresponding min-max values 2.03×10^{-4} and 0.150 given in [2] where the same design problem was considered.

Example 2. This example corresponds to a nearly linear phase bandpass filter with the following specifications

$$D_{BP}(\omega) = \begin{cases} 0 & \omega \in [-1\ \ -0.8] \\ e^{-j10\omega} & \omega \in [-0.6\ \ -0.4] \\ 0 & \omega \in [-0.2\ \ 0] \\ e^{-j8\omega} & \omega \in [0.2\ \ 0.8] \end{cases} \tag{15}$$

$$W(\omega) = \begin{cases} 1 & \text{in passbands} \\ 10 & \text{in stobands} \end{cases} \tag{16}$$

By approximating the above ideal response with a filter of length 25 the resulted maximum approximation errors in magnitude and passband group delays are 0.0552 and 0.7743. These deviations compare favorably with the values 0.037 and 0.6858 given in [7] where the same filter was designed. Fig. 1 depicts the magnitude of the resulting optimum filter. Finally in Fig. 4 we plot the approximated and the desired passband group delays of the filter.

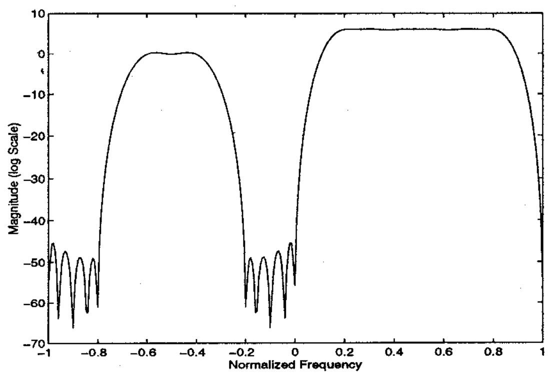

Figure 1. Magnidute response in dB of the nearly linear phase bandpass filter.

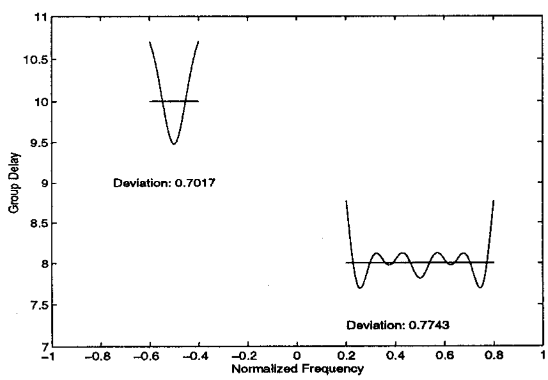

Figure 2. Approximated and desired passband group delays of the nearly linear phase bandpass filter.

4. CONCLUSION

We have presented a new L_2 based method for the design of complex FIR digital filters. By minimizing a suitable L_2 measure we were able to optimally define the part of the ideal response that was not explicitly specified in the design requirements. We have also presented a suboptimal method for the design of weighted filters. The proposed method outperformed the "don't care" method while at the same time compared well with the optimum minmax approximation. The complexity of the proposed method was low because it required the solution of a linear system.

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	Proposed		Don't Care	
$2N+1$	e_m	e_τ	e_m	e_τ
101	1.67×10^{-2}	1.618	3.08×10^{-2}	1.958
151	3.37×10^{-3}	0.539	6.68×10^{-3}	0.919
201	6.00×10^{-4}	0.107	1.40×10^{-3}	0.294
251	1.18×10^{-4}	0.014	3.19×10^{-4}	0.075

Table I. Maximum approximation errors for magnitude and passband group delay for the filter of Example 1.