

Analysis of Discrete-Time Server Queues with Bursty Markovian Inputs

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Abstract

We study a discrete-time deterministic server queue with infinite buffer and with packet arrivals that depend on a multidimensional Markov process. Using the generating functions approach we give a means for obtaining the boundary conditions vector for the case where the value zero is a multiple eigenvalue of the problem. We also derive recursive form expressions for the direct determination of the moments of the queue length. These expressions do not require the knowledge of the steady state probabilities of the combined source-queue process. Finally our method is applied to two queueing problems related to data transmission through an ATM switch with a queue of infinite length. Closed form expressions for the first two moments of the corresponding queue lengths are obtained.

I. INTRODUCTION.

During the last years, special interest is deployed on B-ISDNs, due to their ability to simultaneously transfer data, voice and image. B-ISDNs are mainly implemented through Asynchronous Transfer Mode (ATM) whose basic characteristics are the use of fixed-size packets called "cells" as transfer units, and the identification of each communication unit by a cell header label [1], [2]. Cells may come from many different sources, such as computers, disc units, telephone devices, image transmission units etc. Each cell, in the way to its final destination, is transferred from an ATM switch to another located to a different place. Cells are stored in the ATM switches, waiting for their next transmission. The internal speed of an ATM switch is fast enough, so that we can focus our attention only to its output queue [3].

The output queue of an ATM switch can be considered as a discrete time deterministic server queue, with its time slot equal to the transmission time of a cell. It is clear that the behavior of such a queue depends on its cell arrival process.

The study of the queue length of a discrete -time deterministic server queue, where packet arrivals depend on Markov processes, is of special interest. In [4] an infinite buffer queue is studied with voice packet arrivals characterized by a Markov chain and low priority data message arrivals, characterized by a Poisson process. In [5] the problem for sources that are two or three state Markov chains is considered. Cell arrivals to an output queue of an ATM switch from a GI-stream and from a M-stream are studied in [6]. These streams are considered as Markov-chain driven arrival probabilities. In [7] it is considered cell arrivals coming from a combination of a GI-stream, an M-stream and a B-stream (which is the sum of one-step Markov chains). In [8] the queue behavior is studied under the assumption that in each of the N input trucks, one cell arrives every T deterministic time slots. In [9] the same problem is studied under the extension that each input truck can be active or inactive. In [10] every input truck has a possible different deterministic rate thus for the i -th input truck a cell arrives every d_i time slots.

In this papers we consider two different source models. In the first model sources can decide wether to send a number of cells or not. Once a source decides to send k cells, where k any nonnegative integer, it will be sending cells for the next k time slots. During this time no other decision is possible. When the task is completed the source is again allowed to make decisions. Every cell that is send by the source is directly entering the queue. This model applies, for example, to the problem of transferring files of different lengths through an ATM switch. A similar model was used in [17] but with a basic difference, namely, when the source decided to send k cells the transfer was performed in a single time slot.

The second model we are going to consider is the case where the source decides whether or not to send a given number of k packets. Once the decision to send is made the source will be sending packets for the next

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k time instants. The packets do not enter directly into the queue but rather into an auxiliary buffer devoted to the specific source. Once all k packets have entered the buffer, its content is directly transferred to the main queue. This model applies, for example, to the structure of a normal queue at the physical layer. Normal cells are comprised of a fixed number of bytes (or bits). A cell does not enter the queue unless all bytes of the cell are received at the queue. We give the possibility to each source to have a different length for its cell (in number of bytes).

In order to solve the two problems we just described we focus our attention to the size of the queue length. Using generating functions we obtain a general formula for the stationary probabilities of the combined source-queue process. For the determination of the boundary conditions we follow an approach similar to [13] and extend it to cover our case. A different approach to the boundary conditions problem can be found in [11] where an algorithm, based on the matrix analytic approach of [12], is presented. The second result of this papers consists in the derivation of a general method for recursively obtaining the moments of the queue length. The technique used to derive the recursive expressions is similar to the technique presented in [14]. The method is then applied to the two problems described above and closed form expressions for the mean and variance of the corresponding queue lengths are obtained.

The rest of this paper is organized as follows, in Section II we present the general background and we concentrate our study in determining the boundary conditions for the case of multiple zero eigenvalues. In Section III recursive formulas for the determination of the queue length moments are derived. In Section IV the theory is applied to the two queueing problems. Finally Section V contains the conclusion.

II. GENERAL BACKGROUND.

The queueing model we use is a typical discrete - time, deterministic - server queue with infinite buffer. Time is slotted, and if the queue is not empty, one packet departs from the queue at the beginning of each time slot. In the remaining of the time slot, new packets arrive at the queue from m independent input trucks. Each input truck is driven from a Markov process. Each Markov process enters in a new state at the beginning of each time slot and the probability to send i packets depends on this new state. We consider each Markov process to be independent of the others. The combination of these Markov processes forms a multidimensional Markov process which we will call the *Source Process* or simply "the source". We will also assume that the probability distribution of packet arrivals depends every time on the state of the source process.

Let $q[r]$ denote the state of the source process at the end of the r -th time slot, then let us denote by p_{ij} the state transition probabilities, that is

$$p_{ij} = \mathbf{Pr}[q[r+1] = x_i / q[r] = x_j] \quad (1)$$

where x_0, x_1, \dots, x_K are the possible states of the source process. The state - transition table then takes the form

$$P = \begin{bmatrix} p_{00} & p_{01} & \dots & p_{0K} \\ p_{10} & p_{11} & \dots & p_{1K} \\ \vdots & \vdots & & \vdots \\ p_{K0} & p_{K1} & \dots & p_{KK} \end{bmatrix} \quad (2)$$

The combination of the source and the queue forms a large (actually infinite state) Markov process the stationary probabilities of which we denote by $g_m(n)$, that is, $g_m(n) = \mathbf{Pr}$ [at the end of the time slot, the source process is in state x_m and there are n packets in the queue]. Combining all probabilities that refer to the same queue length in a vector $g(n)$ we have

$$g(n) = [g_0(n) \ g_1(n) \ g_2(n) \ \dots \ g_K(n)]^t \quad (3)$$

The corresponding vector of generating functions $G(z)$ is defined as

$$G(z) = \left[\sum_{n=0}^{\infty} g_0(n)z^n, \sum_{n=0}^{\infty} g_1(n)z^n, \dots, \sum_{n=0}^{\infty} g_K(n)z^n \right]^t \quad (4)$$

Note that $G(1)$ is the stationary source probability vector and thus it is the right normalized eigenvector of P , corresponding to the unit eigenvalue.

At the beginning of the time slot, the source enters at a new state x_j and decides to send l packets with probability $h_j(l)$. Let $h_j(l) = \mathbf{Pr}[l \text{ packets arrive in the queue} / \text{the state of the source is } x_j]$. Denote by $h(l)$ the following diagonal matrix

$$h(l) = \text{diag}[h_0(l), h_1(l), \dots, h_K(l)] \quad (5)$$

and with $H(z)$ that matrix of the corresponding generating functions

$$H(z) = \text{diag} \left[\sum_{l=0}^{\infty} h_0(l)z^l, \sum_{l=0}^{\infty} h_1(l)z^l, \dots, \sum_{l=0}^{\infty} h_K(l)z^l \right] \quad (6)$$

Notice that for $z = 1$ we obtain $H(1) = I$, the unity matrix.

After the above definitions we can easily see that the steady state probability $g_m(n)$ satisfies

$$\begin{aligned} g_m(n) &= \sum_{j=0}^K g_j(n+1)p_{mj}h_m(0) + \sum_{j=0}^K g_j(n)p_{mj}h_m(1) + \dots \\ &+ \sum_{j=0}^K g_j(1)p_{mj}h_m(n) + \sum_{j=0}^K g_j(0)p_{mj}h_m(n) \end{aligned} \quad (7)$$

From Eqs. (3,5,7) we conclude that

$$g(n) = h(0)Pg(n+1) + h(1)Pg(n) + \dots + h(n)Pg(1) + h(n)Pg(0) \quad (8)$$

or using generating function

$$[zI - H(z)P]G(z) = (z-1)H(z)PG(0) \quad (9)$$

Notice that the only unknown quantity in (9) is the vector $G(0)$ of the boundary conditions. Several methods have been proposed for the determination of this vector [11], [13]. In the remaining part of this section we will attempt to give sufficient conditions for determining $G(0)$ for the case where P and $H(z)$ are kronecker products and $z = 0$ is a multiple eigenvalue of the matrix $zI - H(z)P$. Our method will follow the same ideas of [13] but will extend them to the multiple eigenvalue case.

Determination of $G(0)$.

A first equation that is necessary for determining $G(0)$ can be obtained by taking the derivative of (9) with respect to z and then multiplying from the left with the vector $[11 \dots 1]$ [13]. This results in

$$1 - [11 \dots 1]H'(1)G(1) = [11 \dots 1]G(0) \quad (10)$$

Since $[11 \dots 1]G(0) \geq 0$ this sets a constraint on the possible $h_i(l)$ that can be combined with P . If the length of $G(0)$ is $(K+1)$, then we need another K equations for determining $G(0)$. In [13] a method is described which basically consists in obtaining K vectors that are orthogonal to $G(0)$. These vectors are the $f_i(z_i)$, $i = 1, \dots, K$, where $f_i(z)$ is an eigenvector of the matrix $H(z)P$ corresponding to the eigenvalue $\lambda_i(z)$. Also z_i is the solution of the equation $z = \lambda_i(z)$ that lies inside or on the unit circle (except $z = 1$). This theory is valid only for the case where the z_i is simple. In the problems we are going to consider the assumption of simple eigenvalues does not hold. Specifically we will see that the eigenvalue $z = 0$ is multiple and thus the method of [13] is not directly applicable. With the next theorem and the corollaries that follow we will show that, under certain conditions, the result of [13] still applies if the eigenvectors are replaced by the generalized eigenvectors defined in the Jordan Representation Form (JRF). This actually means that $G(0)$ is orthogonal to the whole left eigenspace defined by the eigenvalue $z = 0$.

Before stating the theorem let us first introduce some necessary notations and some elements from Linear Algebra regarding the JRF [15, pp. 364–369]. Let $A_1 \otimes A_2$ denote the kronecker product of the matrices A_1, A_2 and $\otimes_{i=1}^m A_i = A_1 \otimes A_2 \otimes \dots \otimes A_m$. If a matrix A has an eigenvalue λ with multiplicity r then any vector f_0 that satisfies

$$f_0^t(A - \lambda I) = 0 \quad (11)$$

is a left eigenvector of A associated with λ . Let $f_{01}, f_{02}, \dots, f_{0s}$ be a maximal set of linearly independent left eigenvectors associated with λ . Then it is known that $s \leq r$ (geometric dimension no larger than algebraic). When the inequality is strict we need to define generalized eigenvectors in order to obtain the necessary linearly independent vectors that will lead us to the JRF. Notice that we can always select the left eigenvectors in such

a way that they either belong to the range of $A - \lambda I$ or to its orthogonal complement. This is true because a linear combination of eigenvectors is still an eigenvector (when they refer to the same eigenvalue). Thus let f_{01}, \dots, f_{0p} be the eigenvectors that belong to the range of $A - \lambda I$. Each of these eigenvectors initiates a process that generates generalized eigenvectors in the following way

$$\begin{aligned} f_0^t(A - \lambda I) &= 0 && \text{(eigenvector)} \\ f_j^t(A - \lambda I) &= f_{j-1}^t, \quad j \geq 1 && \text{(generalized eigenvector)} \end{aligned} \quad (12)$$

The process is stopped when the first linearly dependent vector is obtained. The index j is the ‘‘order’’ of the generalized eigenvector. The combination of left eigenvectors and generalized eigenvectors of an eigenvalue λ spans the whole left eigenspace that is associated with λ . Notice that a very useful property which is true for the regular eigenvectors is also satisfied by the generalized eigenvectors. Specifically, all (generalized) left eigenvectors of an eigenvalue λ are orthogonal to all (generalized) right eigenvectors of any other eigenvalue. This means that left and right eigenspaces associated with different eigenvalues are orthogonal to each other. We are now ready to state our theorem and two corollaries that will yield the necessary generalization to the method of [13].

Theorem 1. Consider the boundary conditions problem defined by Equ. (9). Let $H(z)P$ be of the form $H(z)P = U(z) \otimes A(z) \otimes V(z)$, where $A(z)$ can be written as $A(z) = A(0) + z^d B(z)$, and $U(z), V(z)$ are square matrix polynomials. If $A(0)$ has a multiple eigenvalue at $\lambda = 0$ then the boundary conditions vector $G(0)$ is orthogonal to any vector ϕ_j that has the form

$$\phi_j = u \otimes f_j \otimes v \quad (13)$$

where u, v are arbitrary vectors and f_j is any (generalized) left eigenvector of order up to $d - 1$ associated with the eigenvalue zero.

Proof. The proof is given in the Appendix. We can now prove the following two corollaries.

Corollary 1. Let $H(z)P = \otimes_{i=1}^m H_i(z)P_i$ with $H_i(z)P_i = H_i(0)P_i + z^{d_i} B_i(z)$ and $H_i(0)P_i$ having a multiple eigenvalue at zero. Then $G(0)$ is orthogonal to any vector w of the form $w = \otimes_{i=1}^m w_i$ where w_i are arbitrary vectors with the only restriction that at least one of the w_i is a (generalized) eigenvector of $H_i(0)P_i$ of order up to $d_i - 1$ associated with the eigenvalue zero.

Proof. Apply Theorem 1 for $U(z) = \otimes_{j=1}^{i-1} (H_j(z)P_j)$, $V(z) = \otimes_{j=i+1}^m (H_j(z)P_j)$, $A(z) = H_i(z)P_i$, $u = \otimes_{j=1}^{i-1} w_j$, $v = \otimes_{j=i+1}^m w_j$.

Corollary 2. If $H(z)P$ can be written as in Corollary 1 and also a) every $H_i(0)P_i$ has a single nonzero eigenvalue λ_i with multiplicity one, b) every sequence of generalized left eigenvectors of the zero eigenvalue has at most $(d_i - 1)$ elements, then the boundary conditions vector $G(0)$ can be written in a kronecker product form $G(0) = a_0 \otimes_{i=1}^m s_i$, where s_i is the right eigenvector of $H_i(0)P_i$ that is associated with its nonzero eigenvalue.

Proof. Let ψ_i, s_i denote the left and right eigenvectors of $H_i(0)P_i$ associated with the nonzero eigenvalue λ_i and $f_{ji}, j = 0, 1, \dots, k_i$ the (generalized) left eigenvectors of the same matrix associated with the multiple zero eigenvalue. The vector s_i is orthogonal to all vectors f_{ji} since it is associated with a different eigenvalue. According to Corollary 1, $G(0)$ is orthogonal to all vectors w of the form $w = \otimes_{i=1}^m w_i$ where at least one w_i is one of the vectors f_{ji} . Selecting now w_i to be either ψ_i or any f_{ji} with the only constraint that at least one of the w_i to be different than ψ_i we conclude, because of Corollary 2, that $G(0)$ is orthogonal to all these vectors. The number of the vectors we just defined is $(k_1 + 1)(k_2 + 1) \cdots (k_m + 1) - 1$ (all possible combinations except the one where all w_i equal ψ_i). $G(0)$ being of size $(k_1 + 1)(k_2 + 1) \cdots (k_m + 1)$ is thus uniquely defined (modulo a multiplicative constant). Consequently if we set

$$G(0) = a_0 \otimes_{i=1}^m s_i \quad (14)$$

we can easily see that this vector satisfies all orthogonality constraints and thus is the vector we are looking for. In order to facilitate certain derivations later in the paper, without loss of generality, we assume for each s_i that it satisfies

$$[1 \ 1 \ \cdots \ 1] s_i = 1 \quad (15)$$

and thus from (10) we determine a_0 as

$$a_0 = 1 - [1 \ 1 \ \cdots \ 1]H'(1)G(1) \quad (16)$$

III. MOMENT ANALYSIS.

In this section we will present a general method for obtaining directly the moments of the queue length without the need of finding the stationary probabilities first. Specifically we will obtain the quantities

$$v_m = [1 \ 1 \ \cdots \ 1]G^{(m)}(z)|_{z=1} \quad (17)$$

where $G^{(m)}$ denotes the m -th derivative of $G(z)$ with $G(z)$ defined in (9). Notice that in order to obtain $G(z)$ we need to compute the inverse of $zI - H(z)P$. This is not always possible since the matrix depends on z . If instead we are only interested in the moments v_m defined in (17) then it is possible, most of the time, to compute these moments by solving small linear systems with constant coefficients. Let us consider this problem for a slightly more general case. Let us assume $F(z)$ to satisfy the following equation

$$Q(z)F(z) = (z - 1)R(z)F \quad (18)$$

where $Q(z), R(z)$ are square matrices. From (18) we conclude that $Q(1)F(1) = 0$ which means that $Q(1)$ is singular and $F(1)$ is its right eigenvector associated with the zero eigenvalue. Let now x_0 be the left eigenvector i.e. $x_0^t Q(1) = 0$, then we select x_0 and $F(1)$ to satisfy $x_0^t F(1) = 1$. Notice that Equ. (9) is a special case of (18) with $F(z) = G(z)$, $Q(z) = zI - H(z)P$, $R(z) = H(z)P$, $F = G(0)$ and $x_0 = [1 \ 1 \ \cdots \ 1]^t$. We are now interested in obtaining

$$u_m = x_0^t F^{(m)}(z)|_{z=1} \quad (19)$$

The following theorem describes a recursive method for the computation of u_m . Analogous recursive expressions for the computation of the moments of traffic processes associated to a markovian queueing system have been presented in [14].

Theorem 2. Let u_m be defined as in Equ. (19) then it can be computed via the following recursion

$$u_m = \frac{1}{a_0} \left\{ \sum_{k=0}^m \binom{m}{k} x_k^t R^{(m-k)}(1)F - \frac{1}{m+1} \sum_{k=1}^m \binom{m+1}{k+1} a_k u_{m-k} \right\} \quad (20)$$

where the scalars a_m and the vectors x_m are recursively defined by

$$a_m = \sum_{k=0}^m \binom{m+1}{k} x_k^t Q^{(m+1-k)}(1)F(1) \quad (21)$$

$$x_m^t Q(1) = a_{m-1} x_0^t - \sum_{k=0}^{m-1} \binom{m}{k} x_k^t Q^{(m-k)}(1) \quad (22)$$

Proof. The proof is given in the Appendix.

Notice that the linear system in (22) that defines the vector x_m has an infinity of solutions (because the matrix $Q(1)$ is singular). This property is particularly useful since by selecting a specific solution it is possible to simplify certain expressions as we will shortly see.

If we apply the results of Theorem 2 to $G(z)$ of (9) and use the fact that for this case $Q(z) = zI - H(z)P$, $R(z) = H(z)P$ then Equ. (22) yields

$$\sum_{k=0}^m \binom{m}{k} x_k^t H^{(m-k)}(1)PG(0) = x_m^t G(0) + m x_{m-1}^t G(0) - a_{m-1} x_0^t G(0) \quad (23)$$

Since the vectors x_k are not uniquely defined ($x_k + \mu x_0$ also satisfies (22)) we impose as constraint on x_k to be orthogonal to $G(0)$, that is

$$x_k^t G(0) = 0, \quad k = 1, 2, \dots \quad (24)$$

This, using Eqs. (16,20), finally yields

$$v_m = -a_{m-1} - \frac{1}{(m+1)a_0} \sum_{k=1}^m \binom{m+1}{k+1} a_k v_{m-k}, \quad v_0 = 1 \quad (25)$$

where

$$a_m = (m+1)x_m^t G(1) - \sum_{k=0}^m \binom{m+1}{k} x_k^t H^{(m+1-k)}(1)G(1) \quad (26)$$

$$x_m^t [I - P] = a_{m-1}x_0^t - mx_{m-1}^t + \sum_{k=0}^{m-1} \binom{m}{k} x_k^t H^{(m-k)}(1)P, \quad x_m^t G(0) = 0 \quad (27)$$

In the rest of the section we will apply these results to obtain expressions for the mean and variance of a queue whose source process consists of kronecker products of smaller processes.

Computation of the Mean and Variance for Kronecker Products.

We refer again to Equ. (9). For the mean and the variance, applying Eqs. (25,26,27), we obtain

$$\begin{aligned} v_1 &= -a_0 - \frac{a_1}{2a_0} \\ \sigma^2 &= \frac{a_1^2}{4a_0^2} - a_0^2 - a_1 - \frac{a_2}{3a_0} \end{aligned} \quad (28)$$

where a_0, a_1, a_2 are defined by

$$\begin{aligned} a_0 &= 1 - x_0^t H'(1)G(1) \\ a_1 &= 2x_1^t G(1) - x_0^t H''(1)G(1) - 2x_1^t H'(1)G(1) \\ a_2 &= 3x_2^t G(1) - x_0^t H'''(1)G(1) - 3x_1^t H''(1)G(1) - 3x_2^t H'(1)G(1) \end{aligned} \quad (29)$$

and x_1, x_2 by

$$\begin{aligned} x_1^t [I - P] &= a_0 x_0^t - x_0^t + x_0^t H'(1)P, \quad x_1^t G(0) = 0 \\ x_2^t [I - P] &= a_1 x_0^t - 2x_1^t + x_0^t H''(1)P + 2x_1^t H'(1)P, \quad x_2^t G(0) = 0 \end{aligned} \quad (30)$$

From a computational point of view it is the equations in (30) that are the heaviest because they require the solution of linear systems. Unfortunately, in most practical situations these two linear system are very large. This is particularly the case when the matrices $H(z)$ and P are kronecker products of smaller matrices. We will now present a method for computing x_1 and x_2 by solving a number of problems of the form of (30) but that are of the size of the matrices that constitute H and P .

Let us assume the following kronecker form for the matrices of interest

$$\begin{aligned} P &= \otimes_{i=1}^m P_i, \quad H(z) = \otimes_{i=1}^m H_i(z), \quad I = \otimes_{i=1}^m I_i \\ x_0 &= \otimes_{i=1}^m x_{0i}, \quad G(1) = \otimes_{i=1}^m G_i(1) \end{aligned} \quad (31)$$

We also need the first two derivatives of the matrix $H(z)$ at $z = 1$. They satisfy the expressions

$$\begin{aligned} H'(1) &= \sum_{i=1}^m (\otimes_{j=1}^{i-1} I_j) \otimes H'_i(1) \otimes (\otimes_{j=i}^m I_j) \\ H''(1) &= \sum_{i=1}^m (\otimes_{j=1}^{i-1} I_j) \otimes H''_i(1) \otimes (\otimes_{j=i}^m I_j) \\ &\quad + 2 \sum_{i=1}^m \sum_{n=i+1}^m (\otimes_{j=1}^{i-1} I_j) \otimes H'_i(1) \otimes (\otimes_{j=i}^{n-1} I_j) \otimes H'_n(1) \otimes (\otimes_{j=n}^m I_j) \end{aligned} \quad (32)$$

We can then show by direct application that the vectors x_1, x_2 can take the form

$$\begin{aligned} x_1 &= \sum_{i=1}^m (\otimes_{j=1}^{i-1} x_{0j}) \otimes x_{1i} \otimes (\otimes_{j=i+1}^m x_{0j}) \\ x_2 &= \sum_{i=1}^m (\otimes_{j=1}^{i-1} x_{0j}) \otimes x_{2i} \otimes (\otimes_{j=i+1}^m x_{0j}) \\ &\quad + 2 \sum_{i=1}^m \sum_{j=i+1}^m (\otimes_{n=1}^{i-1} x_{0n}) \otimes x_{1i} \otimes (\otimes_{n=i+1}^{j-1} x_{0n}) \otimes x_{1j} \otimes (\otimes_{n=j+1}^m x_{0n}) \end{aligned} \quad (33)$$

where the vectors x_{1i}, x_{2i} satisfy

$$\begin{aligned} x_{1i}^t [I_i - P_i] &= a_{0i} x_{0i}^t - x_{0i}^t + x_{0i}^t H_i'(1) P_i \\ x_{2i}^t [I_i - P_i] &= a_{1i} x_{0i}^t - 2(a_0 - a_{0i} + 1) x_{1i}^t + x_{0i}^t H_i''(1) P_i + 2x_{1i}^t H_i'(1) P_i \end{aligned} \quad (34)$$

with

$$\begin{aligned} a_{0i} &= 1 - x_{0i}^t H_i'(1) G(1) \\ a_{1i} &= 2(a_0 - a_{0i} + 1) x_{1i}^t G_i(1) - x_{0i}^t H_i''(1) G_i(1) - 2x_{1i}^t H_i'(1) G_i(1) \\ a_{2i} &= 3(a_0 - a_{0i} + 1) x_{2i}^t G_i(1) - x_{0i}^t H_i'''(1) G_i(1) - 3x_{1i}^t H_i''(1) G_i(1) - 3x_{2i}^t H_i'(1) G_i(1) \\ &\quad - 3(a_0 - a_{0i}) a_{1i} + 3 \left(2(1 - a_{0i}) [x_{1i}^t G_i(1) - a_{0i}] + a_{1i} \right) \left(\sum_{j=1}^m x_{1j}^t G_j(1) - x_{1i}^t G_i(1) \right) \end{aligned} \quad (35)$$

Again the linear systems in (34) have an infinite number of solutions. For the case where $G(0)$ is a kronecker product (as is the case of Corollary 2) and $G(0)$ can take the form of Equ. (14) we can require for every i

$$x_{1i}^t s_i = x_{2i}^t s_i = 0 \quad (36)$$

This will be sufficient for the validity of Equ. (24) for $k = 1, 2$ and consequently for the validity of the expressions for the mean and variance in (28).

Let us now relate a_0 and a_1 to their small problem counterparts. Using the form of x_0, x_1 from Equ. (33) we conclude after some algebra that

$$\begin{aligned} a_0 &= 1 - \sum_{i=1}^m (1 - a_{0i}) \\ a_1 &= \sum_{i=1}^m a_{1i} + \sum_{i=1}^m (1 - a_{0i})^2 - (1 - a_0)^2 \\ a_2 &= \sum_{i=1}^m a_{2i} - 2 \sum_{i=1}^m (1 - a_{0i})^3 + 3(1 - a_0) \sum_{i=1}^m (1 - a_{0i})^2 - (1 - a_0)^3 \end{aligned} \quad (37)$$

In the next sections, where we present the applications, we will only define a_{0i}, a_{1i}, a_{2i} which combined with (28, 37) can lead to the computation of the mean and variance.

IV. APPLICATIONS.

In this section we are going to apply the results of Sections II and III to the two queueing problems introduced in Section I.

Application 1.

We are now going to study in cell level, a deterministic server, discrete time queue where cell arrivals are generated from m independent markovian processes that play the role of cell sources, with the following characteristics: Source i , at the beginning of the each time slot, with a certain probability p_{ki} decides if it will send a burst of k cells, or if it will suspend the decision whether to send or not, for the following time slot. Once a source decides to send a burst of length k , it will be sending cells for the next k time slots, one cell per time slot. Cells immediately enter into the queue, without being accumulated in an auxiliary buffer. The theoretical model of each source, describes for example a the transmission of files consisting of any number of cells, through an ATM network.

Let us denote by p_{ki} the probability that source i decides to send a burst of k cells and by p_{0i} the probability to suspend the decision for the next time slot. Notice that if we like to put this process under a Markov model we need to define auxiliary states that denote the intermediate states of the source when it is sending cells. Thus the state transition table will contain states that correspond the source being in the middle of a sending process and to states where a task of an earlier decision is completed. Notice that the source can make a new

decision only when it has completed an earlier task. Thus the transition table takes the form

$$P_i = \begin{bmatrix} p_{0i} & p_{0i} & 0 & p_{0i} & 0 & 0 & p_{0i} & & \\ p_{1i} & p_{1i} & 0 & p_{1i} & 0 & 0 & p_{1i} & & \\ p_{2i} & p_{2i} & 0 & p_{2i} & 0 & 0 & p_{2i} & & \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & \cdots & \\ p_{3i} & p_{3i} & 0 & p_{3i} & 0 & 0 & p_{3i} & & \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & & \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & & \\ & & & & \vdots & & & & \end{bmatrix} \quad (38)$$

Let us explain the meaning of each state. State 1: delay new decision, State 2: previously decided to send 1 cell, 1 cell is send (task completed), State 3: previously decided to send 2 cells, 1 cell is send, State 4: previously decided to send 2 cells, 2 cells are send (task completed), State 5: previously decided to send 3 cells, 1 cell is send, etc. The unities in the columns indicate that if the source is in an intermediate state it can only go to the next intermediate state (i.e. continue sending) until a previously decided task is completed.

Since the source always sends a cell to the queue, except when it is in the first state, we conclude that

$$H_i(z) = \text{diag}[1 \ z \ z \ \cdots] \quad (39)$$

The vector $G_i(1)$ is

$$G_i(1) = \frac{1}{p_{0i} + \bar{k}_{1i}} [p_{0i}, p_{1i}, p_{2i}, p_{2i}, p_{3i}, p_{3i}, p_{3i}, \dots]^t \quad (40)$$

where \bar{k}_{li} denotes

$$\bar{k}_{li} = \sum_{n=l}^{\infty} n(n-1) \cdots (n-l+1) p_{ni} \quad (41)$$

To find $G(0)$ we apply Corollary 2. Notice that we can write

$$H_i(z)P_i = H_i(0)P + zB_i \quad (42)$$

The matrix $H_i(0)$ is of rank one thus the same will hold for $H_i(0)P_i$. This means that $H_i(0)P_i$ has a single nonzero eigenvalue which is simple. Also, since for any vector of length k there exist $k-1$ linearly independent vectors that are orthogonal to it, we conclude that the matrix $H_i(0)P_i$ has only regular eigenvectors for the eigenvalue zero. Consequently we can apply Corollary 2 and we can easily see that

$$s_i = [1 \ 0 \ 0 \ \cdots]^t \quad (43)$$

Actually for this problem it is very easy to find $G(0)$ since the only case where it is possible to have no cells in the queue is when all sources are in state "0". The vectors x_{1i}, x_{2i} have the form

$$\begin{aligned} x_{1i} &= a_{0i} [0, 0, 1, 0, 2, 1, 0, 3, 2, 1, 0, \dots]^t \\ x_{2i} &= a_{1i} [0, 0, 1, 0, 2, 1, 0, 3, 2, 1, 0, \dots]^t \\ &\quad - a_{0i}(a_0 - a_{0i} + 1) [0, 0, 2 \cdot 1, 0, 3 \cdot 2, 2 \cdot 1, 0, 4 \cdot 3, 3 \cdot 2, 2 \cdot 1, 0, \dots]^t \\ &\quad + a_{0i} [0, 0, 0, 0, 2 \cdot 1, 0, 0, 3 \cdot 2, 2 \cdot 1, 0, 0, 4 \cdot 3, 3 \cdot 2, 2 \cdot 1, 0, 0, \dots]^t \end{aligned} \quad (44)$$

also

$$x_{1i}^t G_i(1) = \frac{p_{0i} \bar{k}_{2i}}{2(p_{0i} + \bar{k}_{1i})^2} \quad (45)$$

and

$$\begin{aligned} a_{0i} &= \frac{p_{0i}}{p_{0i} + \bar{k}_{1i}} \\ a_{1i} &= \frac{p_{0i}}{(p_{0i} + \bar{k}_{1i})^2} (a_0 - a_{0i}) \bar{k}_{2i} \\ a_{2i} &= \frac{a_0 - a_{0i}}{2(p_{0i} + \bar{k}_{1i})} \left(3a_{1i} [\bar{k}_{2i} - 2(p_{0i} + \bar{k}_{1i})] + 2a_{0i} \bar{k}_{3i} - 2a_{0i}(a_0 - a_{0i} + 1)(\bar{k}_{3i} + 3\bar{k}_{2i}) \right) \\ &\quad + 3 \left(2(1 - a_{0i})(x_{1i}^t G_i(1) - a_{0i}) + a_{1i} \right) \left(\sum_{j=1}^m x_{1j}^t G_j(1) - x_{1i}^t G_i(1) \right) \end{aligned} \quad (46)$$

Application 2.

In this application we examine a deterministic-server, discrete-time queue that receives packets originating from m independent Markov processes that play the role of packet sources. Each source has the following characteristics: at the beginning of each time slot it decides with some probability whether to send a burst of k_i packets to the queue or to suspend the decision for the next time slot. Once it decides to send, it will be sending one packet per time slot, for the next k_i time slots. When the source completes the sending stage it can again make a new decision. This model resembles to the model of the previous application, only now each source can send a burst of a specific length. Also packets originating from the same source are accumulated into an auxiliary buffer waiting the arrival of the k_i -th packet. Upon its arrival, the whole collection of packets enters into the queue. If more than one bursts of packets are ready to simultaneously enter the queue, they enter in random order or according to predetermined priorities.

Using this model, we can find the moments of the output queue of an ATM switch in the physical layer. Packets (bytes or bits) that constitute a cell enter into an auxiliary buffer until the whole cell is received and then enter into the output queue. Each cell has a fixed size length, so every source sends a fixed number of k packets to the queue.

The model can also be used for determining the moments of the main buffer in the RARES parallel database machine [16]. In this machine, there is a R/W head per disc track, with a built-in comparison circuit used to examine and quickly decide whether a tuple satisfies the searching criteria set by the main processor. Tuples are stored on the disc in such a way, that various R/W heads can work in parallel. Tuples belonging to the same (different) relation have the same (generally different) length. A relation can occupy disc space belonging to several R/W heads, and the main processor is able to set searching criteria for more than one relations. So it is possible that more than one R/W heads are in searching status. When a tuple satisfies the searching criteria, the R/W head sends it to the main buffer, where it is stored and from where it is finally transferred to the main processor.

If we denote by k_i the number of packets that source i can send and by p_i the probability that source i suspends the decision for the next time slot, then the state transition table for source i becomes

$$P_i = \begin{bmatrix} p_i & 0 & 0 & \dots & 0 & p_i \\ 1-p_i & 0 & 0 & \dots & 0 & 1-p_i \\ 0 & 1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & 0 \end{bmatrix}_{(k_i+1) \cdot (k_i+1)} \quad (47)$$

and the generating function of the matrix of packet arrivals for source i is:

$$H_i(z) = \text{diag} [1 \ 1 \ \dots \ 1 \ z^{k_i}]_{(k_i+1) \cdot (k_i+1)} \quad (48)$$

The m -th derivative of $H_i(z)$ at $z = 1$ is

$$H_i^{(m)}(1) = k_i(k_i - 1) \dots (k_i - m + 1) \text{diag}[0 \ 0 \ \dots \ 0 \ 1]_{(k_i+1) \cdot (k_i+1)} \quad (49)$$

We can easily obtain $G_i(1)$ which has the form

$$G_i(1) = \delta_i [p_i (1-p_i) (1-p_i) \dots (1-p_i)]^t \quad (50)$$

where

$$\delta_i = \frac{1}{(1-p_i)k_i + p_i} \quad (51)$$

To find $G(0)$ we apply Corollary 2. Notice that $H(z)P$ can be written as $H(z)P = \otimes_{i=1}^m H_i(z)P_i$. Furthermore we have

$$H_i(z)P_i = H_i(0)P_i + z^{k_i} [00 \dots 01]^t [00 \dots 010] \quad (52)$$

The characteristic polynomial of $H_i(0)P_i$ is equal to $\lambda^{k_i}(\lambda - p_i)$. Also, since all generalized eigenvectors have order that cannot exceed k_i , all prerequisites of Corollary 2 are satisfied and we conclude that $G(0)$ is given by (14) where

$$s_i = [p_i^{k_i-1}, (1-p_i)p_i^{k_i-2}, \dots, (1-p_i)p_i, (1-p_i), 0]^t \quad (53)$$

Solving the systems in (34) we have

$$\begin{aligned} x_{1i} &= k_i[0, 1, \dots, 1, 0]^t + k_i\delta_i[0, (k_i - 1), (k_i - 2), \dots, 1, 0]^t + \mu_{1i}x_{0i}^t \\ x_{2i} &= b_i[0, 1, \dots, 1, 0]^t + c_i[0, (k_i - 1), (k_i - 2), \dots, 1, 0]^t \\ &- d_i[0, k_i(k_i - 1), (k_i - 1)(k_i - 2), \dots, 2 \cdot 1, 0]^t + \mu_{2i}x_{0i}^t \end{aligned} \quad (54)$$

where

$$\begin{aligned} b_i &= k_i(k_i - 1 + 2\mu_{1i}) \\ c_i &= a_{1i} - 2(a_0 - a_{0i} + 1)(\mu_{1i} + k_i) \\ d_i &= (a_0 - a_{0i} + 1)k_i\delta_i \\ \mu_{1i} &= k_i(p_i^{k_i-1} - 1) + k_i\delta_i\left(k_i p_i^{k_i-1} - \frac{p_i^k - 1}{p_i - 1}\right) \\ \mu_{2i} &= b_i(p_i^{k_i-1} - 1) + c_i\left(k_i p_i^{k_i-1} - \frac{1 - p_i^{k_i}}{1 - p_i}\right) \\ &- d_i\left(k_i(k_i + 1)p_i^{k_i-1} + 2\frac{(k_i + 1)p_i^{k_i}}{1 - p_i} - 2\frac{1 - p_i^{k_i+1}}{(1 - p_i)^2}\right) \end{aligned} \quad (55)$$

and

$$\begin{aligned} a_{0i} &= p_i\delta_i \\ a_{1i} &= (a_0 - a_{0i} + 1)[2(1 - a_{0i})(k_i - 1) + \delta_i(1 - a_{0i})k_i(k_i - 1)] \\ &- (k_i - 1)(1 - a_{0i}) + 2a_0\mu_{1i} \\ a_{2i} &= 3(a_0 - a_{0i} + 1)[x_{2i}^t G_i(1) - a_{1i}] - (1 - a_{0i})[(k_i - 1)(3\mu_{1i} + k_i - 2) + 3\mu_{2i}] \\ &+ 3\left(2(1 - a_{0i})[x_{1i}^t G_i(1) - a_{0i}] + a_{1i}\right)\left(\sum_{j=1}^m x_{1j}^t G_j(1) - x_{1i}^t G_i(1)\right) + 3a_{1i} \end{aligned} \quad (56)$$

where

$$\begin{aligned} x_{1i}^t G_i(1) &= \frac{1}{2}(1 - a_{0i})(k_i - 1)(2 + \delta_i k_i) + \mu_{1i} \\ x_{2i}^t G_i(1) &= \frac{1}{6}(1 - a_{0i})\left(3c_i(k_i - 1) - 2d_i(k_i^2 - 1) + 6b_i\right) - b_i\delta_i(1 - p_i) + \mu_{2i} \end{aligned} \quad (57)$$

V. CONCLUSION.

In this paper we have presented a method for obtaining, recursively, moments of an infinite length queue that accepts cells emanating from a markovian source. The method is then applied to the case where the markovian source is the kronecker product of smaller markovian processes and formulas for the first two moments are presented that are based on quantities related to the small markovian models. Additionally, a theorem is presented for the determination of the boundary conditions vector which appears in the problem of defining the stationary probabilities of the combined source-queue process. This theorem refers to the case where the value zero is a multiple eigenvalue of the problem and extends existing results that apply to the simple eigenvalues case. The theoretical results are consequently applied to two application problems. The problems that are examined refer to data transmission through an ATM switch. In the first problem the transmission of variable length files (in number of cells) is examined, while in the second, the transmission of cells as collection of smaller units (bytes or bits) is considered. Closed form expressions for the mean and variance of the corresponding queues are obtained.

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Appendix.

Proof of Theorem 1. Consider $G(z)$ from Equ. (9). Notice that $G(z)$, as a power series, must have only nonnegative powers (causal series). If we write $G(z) = G(0) + z\hat{G}(z)$ and define $F(z) = \hat{G}(z) + G(0)$ then we can easily see that $F(z)$ satisfies

$$[zI - H(z)P]F(z) = (z - 1)G(0) \quad (58)$$

where again $F(z)$ corresponds to a causal series. Substituting now in (58) the matrix $H(z)P$ with the assumed form of the theorem, yields

$$\left[zI - U(z) \otimes (A(0) + z^d B(z)) \otimes V(z) \right] F(z) = (z - 1)G(0) \quad (59)$$

We would like to show that for arbitrary vectors u, v and any generalized eigenvector f_j of order $j \leq d - 1$ of the matrix $A(0)$, that $G(0)$ is orthogonal to $u \otimes f_j \otimes v$. Notice that since the (generalized) eigenvectors refer to the zero eigenvalue they satisfy

$$f_0^t A(0) = 0 \quad \text{and} \quad f_j^t A(0) = f_{j-1}^t, \quad j = 1, 2, \dots \quad (60)$$

If we multiply (59) from the left by $\phi_0 = u^t \otimes f_0^t \otimes v^t$ we obtain

$$z \left[u^t \otimes f_0^t \otimes v^t - z^{d-1} (u^t U(z)) \otimes (f_0^t B(z)) \otimes (v^t V(z)) \right] F(z) = (z - 1) \left[u^t \otimes f_0^t \otimes v^t \right] G(0) \quad (61)$$

Notice that the lhs of Equ. (61) corresponds to a causal series with the constant term equal to zero. This must also hold for the rhs, thus we conclude that $G(0)$ must be orthogonal to $u^t \otimes f_0^t \otimes v$. Since we assumed arbitrary u, v the orthogonality property will also hold for vectors that are parametrized by z .

To show now the theorem for f_1 we multiply (59) from the left by $\phi_1^t = zu^t \otimes f_1^t \otimes v^t + (u^t U(z)) \otimes f_0^t \otimes (v^t V(z))$ and this yields

$$\begin{aligned} z^2 \left[u^t \otimes f_1^t \otimes v^t - z^{d-1} (u^t U(z)) \otimes (f_1^t B(z)) \otimes (v^t V(z)) \right. \\ \left. - z^{d-2} (u^t U^2(z)) \otimes (f_0^t B(z)) \otimes (v^t V^2(z)) \right] F(z) = \\ (z - 1) \left[zu^t \otimes f_1^t \otimes v^t + (u^t U(z)) \otimes f_0^t \otimes (v^t V(z)) \right] G(0) \end{aligned} \quad (62)$$

Notice again that the lhs is causal but now the first two terms of the corresponding sequence are zero. The same must hold for the rhs. Since $G(0)$ is orthogonal to the term that contains f_0 we conclude that it must also be orthogonal to the term that contains f_1 . Again since this orthogonality holds for arbitrary vectors u, v it will also hold for vectors that are parametrized by z . Thus in general if we multiply with a vector ϕ_j that has the form

$$\phi_j^t = \sum_{n=0}^j z^{j-n} (u^t U^n(z)) \otimes f_{j-n}^t \otimes (v^t V^n(z)) \quad (63)$$

yields

$$z^{j+1} \left[u^t \otimes f_j^t \otimes v^t - \sum_{n=0}^j z^{d-1-n} \left(u^t U^{n+1}(z) \right) \otimes f_{j-n}^t B(z) \otimes \left(v^t V^{n+1}(z) \right) \right] = (z-1) \phi_j^t G(0) \quad (64)$$

Every time we increase j by one we show the orthogonality for a new generalized eigenvector. This process can continue as long as the quantity in the brackets corresponds to a causal sequence. And this holds for $j \leq d-1$. This concludes the proof.

Proof of Theorem 2. Let us first compute the l -th derivative of the Equ. (18) at $z=1$, this yields

$$\frac{1}{l} \sum_{j=0}^l \binom{l}{j} Q^{(l-j)}(1) F^{(j)}(1) = R^{(l-1)}(1) F \quad (65)$$

Multiplying (65) with $\binom{m}{m+1-l} x_{m+1-l}$ and summing over l , $l=1, \dots, m+1$, we have

$$\begin{aligned} \sum_{l=1}^{m+1} \sum_{j=0}^l \frac{1}{l} \binom{m}{m+1-l} \binom{l}{j} x_{m+1-l}^t Q^{(l-j)}(1) F^{(j)}(1) = \\ \sum_{l=1}^{m+1} \binom{m}{m+1-l} x_{m+1-l}^t R^{(l-1)}(1) F \end{aligned} \quad (66)$$

Changing the order of summation, also changing variables to $n = m+1-l$, $k = m+1-j$ and using the fact that

$$\frac{1}{m+1-n} \binom{m}{n} \binom{m+1-n}{m+1-k} = \frac{1}{m+1} \binom{m+1}{k} \binom{k}{n} \quad (67)$$

after some algebra we obtain

$$\begin{aligned} \frac{1}{m+1} \sum_{k=1}^m \binom{m+1}{k} \left[\sum_{n=0}^k \binom{k}{n} x_n^t Q^{(k-n)}(1) \right] F^{(m+1-k)}(1) \\ + \frac{1}{m+1} \sum_{n=0}^m \binom{m+1}{n} x_n^t Q^{(m+1-n)}(1) F(1) = \sum_{n=0}^m \binom{m}{n} x_n^t R^{(m-n)}(1) F \end{aligned} \quad (68)$$

Notice now that if we require the quantity in the brackets to be equal to $a_{k-1} x_0^t$ where a_{k-1} a scalar, we have

$$\sum_{n=0}^k \binom{k}{n} x_n^t Q^{(k-n)}(1) = a_{k-1} x_0^t \quad (69)$$

Solving for x_k we obtain

$$x_k^t Q(1) = a_{k-1} x_0^t - \sum_{n=0}^{k-1} \binom{k}{n} x_n^t Q^{(k-n)}(1) \quad (70)$$

which is Equ. (22). The parameter a_{k-1} must be selected in order for this equation to have a solution (recall that $Q(1)$ is singular). This is assured by multiplying from the right with $F(1)$, thus yielding zero for the lhs and defining a_{k-1} as in Equ. (21). Finally if we use (69, 70) in (68) and solve for u_m we obtain Equ. (20). This concludes the proof.