A Combinatorial Proof for the Dowry Problem

Xujun Liu
Department of Foundational Mathematics
Xi'an Jiaotong-Liverpool University
Suzhou, China
Email: xujun.liu@xjtlu.edu.cn

Olgica Milenkovic
ECE Department
University of Illinois
Urbana, IL, USA
Email: milenkov@illinois.edu

George V. Moustakides
ECE Department
University of Patras
Rio, Greece
Email: moustaki@upatras.gr

***Abstract*—The Secretary problem is a classical sequential decision-making question that can be succinctly described as follows: a set of rank-ordered applicants are interviewed sequentially for a single position. Once an applicant is interviewed, an immediate and irrevocable decision is made if the person is to be offered the job or not and only applicants observed so far can be used in the decision process. The problem of interest is to identify the stopping rule that maximizes the probability of hiring the highest-ranked applicant. A multiple-choice version of the Secretary problem, known as the Dowry problem, assumes that one is given a fixed integer budget for the total number of selections allowed to choose the best applicant. It has been solved using tools from dynamic programming and optimal stopping theory. We provide the first combinatorial proof for a related new *query-based model* for which we are allowed to solicit the response of an expert to determine if an applicant is optimal. Since the selection criteria differ from those of the Dowry problem, we obtain nonidentical expected stopping times.**

I. Introduction

The classical Secretary problem, also known as the game of googol and the picky bride problem, can be stated as follows: N applicants apply for a single available Secretary position and the set of N applicants can be ranked from the best to worst without ties according to their qualifications for the job. The applicants are presented sequentially and uniformly at random. Once an applicant is interviewed, an immediate decision is made on whether the person is accepted or rejected for the position and the decision cannot be revoked at a later time. Furthermore, only applicants observed so far can be used in the decision process. The problem of interest is to identify the best stopping rule, i.e., the rule that maximizes the probability of hiring the highest ranked applicant.

The Secretary problem was formally introduced by Gardner [7], [8] and is considered a typical example in sequential analysis, optimization, and decision theory. Lindley [13] established the best strategy using algebraic methods while Dynkin [3] independently solved the problem by viewing the selection process as a Markov chain. For N large enough, the solution to the problem turns out to be surprisingly simple: the employer needs to reject the first N/e applicants, where e is the base of the natural logarithm, and then accept the next applicant whose qualification is better than that of all previously observed applicants. The probability of identifying the best applicant tends to $1/e$ as N tends to infinity.

The classical Secretary problem has been generalized in many directions [5], [6], [9], [11], [12], [14], [15], [18], [19], including the Prophet inequality model. One such generalization, the Dowry problem (with multiple choices), introduced by Gilbert and Mosteller [9], assumes that one is given a total of s opportunities to select the best applicant, where $s \geq 1$. Using Dynkin's approach, Sakaguchi [20] rigorously determined the optimal selection times for the Dowry setting.

Motivated by recent works on learning with queries [1], [2], [17], we introduce the problem of query-based sequential selection. In our new model, we assume that the decision making entity has a fixed number of $s-1$ opportunities to query an infallible expert. When an applicant is identified as the potentially highest ranked applicant after an exploration process, the expert provides an answer of the form "the best" or "not the best". If the answer is "the best", then the sequential examination process terminates. However, if the expert responds "not the best", a new exploration-exploitation stage begins as long as the query budget allows it. After the budget is exhausted, one is still allowed to make a final selection without consulting the expert. Therefore, with a budget of $s-1$ queries we can make at most s selections. The goal is to find the strategy that maximizes the probability of selecting the best applicant.

Our new query-based model with the budget of $s-1$ queries is related to but different from the Dowry problem with s selections. For both models, our results indicate that an optimal strategy is a $(a_s, a_{s-1}, \ldots, a_1)$-strategy, i.e., for the i^{th} selection, where $1 \leq i \leq s$ and $1 \leq j = s+1-i \leq s$, we reject the first a_j applicants, wait until the decision for the $(i-1)^{th}$ selection (if $i \geq 2$), and then accept the next applicant whose qualification is better than that of all previously seen applicants. Furthermore, the optimal strategies for the two models with s_1 and s_2 selections in total (w.l.o.g., $s_1 < s_2$) share the same sequence $a_1, a_2, \ldots, a_{s_1}$ when viewed from the right. When $N \to \infty$, our result agrees with the thresholds obtained by Gilbert and Mosteller [9]. On the other hand, the two models are very different from the perspective of the expected termination time, especially when the total number of selections is large.

A combinatorial method for studying the Secretary problem was developed in the works of Fowlkes and Jones [4], and Jones [10], [11]. For the analysis of our query-based model, we adapt the definitions and some ideas from Jones [11], but also introduce a number of new proof techniques needed to solve our more challenging query-based problem. Since the query-based model allows for multiple choices and the optimal strategies differ substantially from the classical ones, our proofs are adapted to accommodate multiple query times.

The paper is organized as follows. Section II introduces the relevant concepts, terminology and models used in the paper. The same section presents technical lemmas needed to establish our main results. Section III describes the optimal selection strategy, while Section IV describes the exact thresholds for the optimal strategy and the maximum probability of identifying the best applicant.

II. Preliminaries

The sample space is the set of all permutations of N elements, i.e. the symmetric group S_N, with the underlying σ-algebra equal to the power set of S_N. The best applicant is indexed by N, the second-best applicant by $N-1,\ldots,$ and the worst applicant by 1. In our model, there is a budget of $s-1$ queries (s selections where $s \geq 1$). A permutation $\pi \in S_N$ is sampled from S_N uniformly at random before the interview process. During the interview process, entries of π are presented one-by-one from the left. The relative ordering of the presented positions of π is the only information that can be used to decide whether to accept the current applicant.

The notion of a prefix is introduced to represent the current relative ordering of applicants. Given a $\pi \in S_N$, the k^{th} prefix of π, denoted by $\pi|_k$, is a permutation in S_k that represents the relabelling of the first k elements of π according to their relative order (e.g., if $\pi = [635124]$, then $\pi|_4 = [4231]$).

Definitions. Let $\sigma \in \bigcup_{i=1}^{N} S_i$ and satisfies $|\sigma| = k$ (length k).

- A permutation $\pi \in S_N$ is said to be σ-*prefixed* if $\pi|_k = \sigma$. For example, $\pi = [165243] \in S_6$ is $\sigma = [1432]$-prefixed.
- A σ-prefixed π is σ-*winnable* if accepting the $|\sigma|^{\text{th}}$ applicant when the prefix σ is presented produces the best applicant for the interview order π. More precisely, for $\sigma = [\sigma(1)\sigma(2)\cdots\sigma(k)]$, we have that π is σ-winnable if π is σ-prefixed and $\pi(k) = N$.
- In a permutation $\pi \in S_N$, a *left-to-right maximum* is a position whose value is larger than all values to the left of the position. For example, if $\pi = [423516] \in S_6$, then the first, fourth, and sixth position are left-to-right maximum.
- A permutation $\sigma \in \bigcup_{i=1}^{N} S_i$ is said to be *eligible* if it ends in a left-to-right maximum *or* has length N (e.g., let $N = 6$; then both $[1324]$ and $[165243]$ are eligible).

Every strategy can be represented as a set of permutations (of possibly different lengths) that lead to an acceptance decision for the last applicant observed; such a set is called a *strike set*. More precisely, the selection process proceeds as follows: if the prefix we have seen so far is in the strike set, then we accept the current applicant and continue (provided a selection remains); if it does not belong to the strike set, we reject the current applicant and continue. For example, let $N = 4$ and $s = 1$. Then, the boxed set of permutations $A = \{[12], [213], [3124], [3214]\}$ in Fig. 1 is a strike set. The corresponding interview strategy may be summarized as follows: if the relative order of the applicants interviewed so far is in the set A, then accept the current applicant; otherwise, reject the current applicant. This turns out to be an optimal strategy with probability of successfully selecting the best applicant equal to $11/24$. Obviously, the strike set representing an optimal strategy only contains eligible permutations, since an optimal strategy only selects applicants that are left-to-right maximum. We also make use of s-strike sets defined below.

Definition. A set $X \subseteq \bigcup_{j=1}^{N} S_j$ is called an s-*minimal set* if it is impossible to have $s+1$ elements $\alpha_1, \alpha_2, \ldots, \alpha_{s+1} \in X$ such that α_{i+1} is a prefix of α_i, for all $i \in \{1, 2, \ldots, s\}$.

Definition. A set of permutations $A \subseteq \bigcup_{j=1}^{N} S_j$ is called an s-*strike set* if it satisfies the following three conditions:

- It comprises prefixes that are eligible.
- The set A is s-minimal. Note that the set A may contain elements $\alpha_1, \alpha_2, \ldots, \alpha_s$ such that α_{i+1} is a prefix of α_i, for all $i \in \{1, 2, \ldots, s-1\}$. In other words, based on an s-strike set one can make at most s selections.
- Every permutation in S_N contains some element of A as its prefix (i.e., given an s-strike set one can always make a selection based on its elements).

From the previous definition and the fact that we are allowed to make at most s selections it follows that any optimal strategy for our problem can be represented by an s-strike set. For example, the set $\{[1], [12], [213], [3124], [3214]\}$ in Fig. 1 is a 2-strike set, which also represents an optimal strategy for the case $N = 4$ and $s = 2$. See also Example 1. Furthermore, for a permutation σ of length k, where $1 \leq k \leq N$, and $i \in \{1, 2, \ldots, s\}$, we make use of the following probabilities:

$Q_i(\sigma)$: The probability of identifying the best applicant with the strategy accepting the k^{th} position and using the best strategy thereafter **conditioned on** the pre-selected interviewing order π being σ-prefixed and i selections still being available when interviewing the applicant at position k.

$Q_i^o(\sigma)$: The probability of identifying the best applicant with the best strategy after making a decision for the k^{th} position **conditioned on** the pre-selected interviewing order π being σ-prefixed and i selections still being available right after the interview of the k^{th} applicant.

$\bar{Q}_i(\sigma)$: $\bar{Q}_i(\sigma) = \max\{Q_i(\sigma), Q_i^o(\sigma)\}$.

Intuitively, Q represents the probability of winning by accepting the current applicant while Q^o is the probability of winning based on future selections in the interview process. In order to ensure the maximum probability of winning, an optimal strategy will examine two available choices, i.e. "accept the current applicant" or "reject the current applicant and implement the best strategy in the future" at each stage of the interview and select the one with a better chance of identifying the best applicant.

The *standard denominator* of a permutation σ, denoted by $\mathrm{SD}(\sigma)$, is the number of σ-prefixed permutations $\pi \in S_N$. Denote the number of σ-winnable permutations $\pi \in S_N$ by $\mathrm{Win}(\sigma)$. The $\oplus$ operation for $\frac{a}{b}$ and $\frac{c}{d}$ is defined as $\frac{a}{b} \oplus \frac{c}{d} = \frac{a+c}{b+d}$. It is used to compute the probability of the union of two disjoint events from two disjoint sample spaces over a new sample space equal to the union of the sample spaces.

For each σ of length $\ell-1$, where $2 \leq \ell \leq N$, we define $f_j(\sigma)$, $1 \leq j \leq \ell$, to be the σ-prefixed permutation of length ℓ such that its last position has value j after relabelling according to the first $\ell-1$ positions of σ. For example, for $\sigma = [123]$, a permutation of length 3, we have $f_1(\sigma) = [2341], f_2(\sigma) = [1342], f_3(\sigma) = [1243]$ and $f_4(\sigma) = [1234]$.

Let σ be a permutation of length $1 \leq k \leq N$ with $Q_i(\sigma), Q_i^o(\sigma), \bar{Q}_i(\sigma)$ defined as above with $1 \leq i \leq s$ selections available right before processing the k^{th} applicant of a σ-prefixed permutation. For each $1 \leq i \leq s$, if the k^{th} position of σ is selected, then the number of selections available decrease by one; if the k^{th} applicant is rejected, then the number of selections available does not change. When the number of available selections becomes zero or all applicants are examined, the process terminates.

Remark 1. *Note that we do not simplify the fractions in the expressions for the probabilities $Q, Q^o, \bar{Q}$ until we solve the problem (for example, if the numerator and denominator of $Q_1(\sigma) = Win(\sigma)/SD(\sigma)$ have a common divisor d, we do not cancel d). After making a decision on the $|\sigma|^{th}$ applicant, the interviewer examines the next applicant while the relative order of the interviewed applicants changes to one of $f_1(\sigma), \ldots, f_{k+1}(\sigma)$. An optimal strategy involves making a decision with the largest probability of winning when encountering each of the $f_1(\sigma), \ldots, f_{k+1}(\sigma)$. Thus,*

$$Q_i^o(\sigma) = \bar{Q}_i(f_1(\sigma)) \oplus \cdots \oplus \bar{Q}_i(f_{k+1}(\sigma)). \tag{1}$$

By the definition of Q_i, Q_i^o, $i \in \{2, \ldots, s\}$, we have

$$Q_i(\sigma) = Q_1(\sigma) + Q_{i-1}^o(\sigma). \tag{2}$$

Eq. (2) holds since there are two (disjoint) events that ensure winning after examining the current applicant, i.e., (a) the current applicant is the best and (b) the current applicant is not the best but we identify the best applicant at a later time with a best strategy after rejecting the current applicant. In Case (a), the probability of successfully identifying the best applicant is $Q_1(\sigma)$; in Case (b), the number of available selections decreases by 1 and the corresponding probability is $Q_{i-1}^o(\sigma)$.

Following a methodology suggested by Jones [11], in our proof we make extensive use of *prefix trees* which naturally capture the inclusion-relationship between prefixes of permutations. The concept is best described by an illustrative example, depicted for the case of S_4 in Fig. 1. The correspondence between sub-trees/sub-forests is crucial for the proof of Lemma 1. For example, let F_1 be the sub-forest obtained by deleting the vertex $[12]$ in the tree induced by $[12]$ and its children, let F_2 be the sub-forest obtained by deleting the vertex $[21]$ in the tree induced by $[21]$ and its children. Then, there is a bijection between F_1 and F_2 which preserves all the probabilities used in evaluating optimal strategies.

For V equal to the collection of all permutations of length at most N, we let $T = (V, E)$ be a graph constructed as follows: if $\sigma, \tau \in V$ and σ is a prefix of τ with $|\sigma| = |\tau| - 1$, then we have an edge $\sigma\tau \in E$.

For each $i \in \{1, \ldots, s\}$, a prefix σ is said to be *type i-positive* if $Q_i(\sigma) \geq Q_i^o(\sigma)$ and *type i-negative* otherwise.

We note that the probabilities $Q_i^o, Q_i, \bar{Q}_i$ for each $i \in \{1, \ldots, s\}$ can be pre-computed using backward recursions. Upon running these recursions, we find the winning probability in Section IV by solving another recurrence relation. Given the probabilities $Q_i^o, Q_i, \bar{Q}_i$ for all permutations and $i \in \{1, \ldots, s\}$, we describe next a procedure for finding an optimal strategy and its corresponding strike set.

Theorem 1. *There exists a strike set A which can be partitioned as $A_s \cup \cdots \cup A_1$, where each A_i is a set of type i-positive 1-minimal permutations, for $1 \leq i \leq s$, so that the maximum probability of winning equals $\oplus_{\sigma \in A_s} Q_s(\sigma)$. Furthermore, the maximum probability of winning equals*

$$\sum_{\sigma \in A} Q_1(\sigma) \cdot \mathrm{SD}(\sigma) \Big/ (N!).$$

Proof. We only provide a sketch of the proof. The optimal winning probability is equal to $\bar{Q}_s([1])$, which is the max of $Q_s([1])$ and $Q_s^o([1])$. We introduce the following algorithm to describe a general approach for obtaining $Q_i^o(\sigma)$. To initialize the algorithm, let $\Gamma_i = \varnothing$ and $B = \{f_1(\sigma), \ldots, f_{\ell+1}(\sigma)\}$. We repeat the "main step"; until the process terminates.

Main step: Check if $B = \varnothing$; if yes, stop and return the set Γ_i; if no, then do the following: pick an arbitrary permutation $\phi \in B$, say of length q, with $|\sigma| < q \leq N$; check if ϕ is both eligible and type i-positive ($Q_i(\phi) \geq Q_i^o(\phi)$); if yes, set $\Gamma_i = \Gamma_i \cup \phi$ and $B = B - \phi$; if no, do not update Γ_i and let $B = (B - \phi) \cup \bigcup_{j=1}^{q+1} f_j(\phi)$.

From the above algorithm we have $Q_i^o(\sigma) = \oplus_{\mu \in \Gamma_i} Q_i(\mu)$, and by (2), $Q_i(\mu) = Q_1(\mu) + Q_{i-1}^o(\mu)$ if $i \geq 2$. Thus, $\bar{Q}_s([1])$ can be derived using an inductive argument. □

The next lemma establishes the useful properties of the probabilities $Q_i(\sigma)$, $Q_i^o(\sigma)$, $\bar{Q}_i(\sigma)$ that they only depend on the length and the value of the last position of σ. The proof uses the correspondence between prefix trees under permutations of the same length. Moreover, the result shows that $Q_i^o(\sigma)$ only depends on the length of σ and does not depend on the value of the last position of σ.

Lemma 1. *For all $1 \leq i \leq s$, the probability $Q_i(\sigma)$ only depends on the length and the value of the last position of σ; at the same time, $Q_i^o(\sigma)$ only depends on the length of σ.*

We see in Fig. 1 that all permutations of the same length (i.e., at the same level of the tree) share the same probabilities Q_1^o and Q_2^o, respectively. Moreover, $Q_1([123]) = Q_1([213])$ and $Q_2([123]) = Q_2([213])$, as they are both eligible and have length 3. In order to simplify our exposition, we henceforth change the notation and let $Q_i(\sigma)$, $Q_i^o(\sigma)$, $\bar{Q}_i(\sigma)$ stand for the *numerators* in the definition of the underlying probabilities, each with respect to denominator $SD(\sigma)$.

We show in Lemma 2 that if an eligible permutation is negative then all eligible permutations of shorter length are negative as well. Define $\bar{Q}_0 = 0$ for any permutation.

Lemma 2. *For $\sigma = [12 \cdots (k-1)]$ and $1 \leq i \leq s$, we have*

$$Q_i^o(\sigma) = (k-1) \cdot Q_i^o(f_k(\sigma)) + \bar{Q}_i(f_k(\sigma)) \quad \textit{and}$$
$$Q_i(\sigma) = (k-1) \cdot Q_i(f_k(\sigma)) + \bar{Q}_{i-1}(f_k(\sigma)).$$

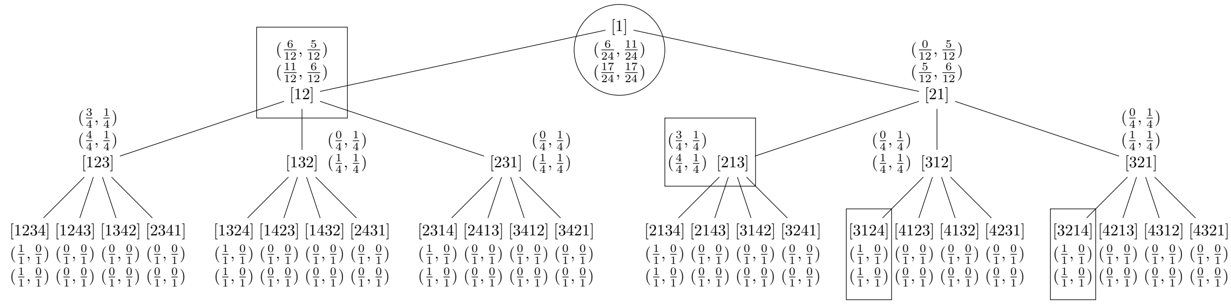

Fig. 1. The prefix tree, the Q_1, Q_1^o, Q_2, Q_2^o probabilities, and a 2-strike set for our problem with four applicants.

Corollary 1. *For increasing permutation $\sigma = [12\cdots(k-1)]$ and $f_k(\sigma) = [12\cdots k]$, we have that if $f_k(\sigma)$ is type i-negative then σ is type i-negative, where $1 \le i \le s$.*

In words, Corollary 1 asserts that each $Q_i(k) - Q_i^o(k)$ is a non-decreasing function of k.

Example 1. To clarify the above concepts, we present an example for the case $s = 2$ and $N = 4$. An optimal strategy is the $(0, 1)$-strategy where we accept the first applicant, ask the expert whether this applicant is the best, and then accept the next left-to-right maximum. The optimal winning probability is $17/24$, an improvement of $6/24$ when compared to the optimal winning probability (equal to $11/24$) for the case when only one selection is allowed (see Fig. 1). Note that for each permutation $\sigma \in S_4$, we list the probabilities Q_1, Q_1^o in the first line and the probabilities Q_2, Q_2^o in the second line underneath each permutation shown in Fig. 1.

III. The optimal strategy

The maximum probability of winning for the Dowry model with s selections and the query-based model with $s-1$ queries are the same, as both models have a budget of s selections and the goal is to choose the best applicant. However, the expected stopping times are very different. Under the query-based model, the process immediately terminates after obtaining a positive answer from the expert. On the other hand, the decision making entity continues to interview the remaining applicants after a selection is made (provided there is a selection left) under the Dowry setting, as it has no information weather the current applicant is the best.

To obtain the optimal strategy for the Dowry model, we observe that $Q_i^o(\sigma) \ge Q_{i-1}^o(\sigma)$, $Q_i(\sigma) \ge Q_{i-1}(\sigma)$, and $Q_i(\sigma) \ge Q_{i-1}^o(\sigma)$ hold true for every $\sigma \in \bigcup_{k=1}^{N} S_k$ and $1 \le i \le s-1$. Let $Q_i(k)$ denote the probability Q_i of eligible prefixes of length k, where $1 \le i \le s$. The probabilities $Q_i^o(k)$ and $\bar{Q}_i(k)$, where $1 \le i \le s$, are defined similarly.

For the i^{th} selection ($j = s+1-i$ selections left), $1 \le i \le s$, by the algorithm described in Theorem 1, we check if σ is eligible and type j-positive (i.e., if $Q_j(\sigma) \ge Q_j^o(\sigma)$); if yes, we accept the current applicant and continue to the next selection (if one is left); if no, we reject the current applicant and continue our search; if there are no selections left, we terminate the process. By Corollary 1, we know each $Q_j(k) - Q_j^o(k)$ is a non-decreasing function of k, which allows us to formulate the optimal strategy.

Theorem 2. *An optimal strategy for the problem with s selections is a positional s-threshold strategy, i.e. there are s numbers $0 \le k_1 \le k_2 \le \ldots \le k_s \le N$ such that when considering the i^{th} selection, where $1 \le i \le s$, we reject the first k_i applicants, wait for the $(i-1)^{th}$ selection, and then accept the next left-to-right maximum.*

Proof. Let $j = s+1-i$. By Corollary 1, and since we know $Q_j(\sigma) \ge Q_j^o(\sigma)$ for a permutation, there exists some $0 \le k_i \le N$ such that $Q_j(k) \ge Q_j^o(k)$ for $k \ge k_i + 1$ and $Q_j(k) < Q_j^o(k)$ for all $k \le k_i$, where $1 \le i \le s$. Therefore, an optimal strategy is to reject the first k_i applicants and then accept the next left-to-right maximum thereafter. It is also clear that every optimal strategy needs to proceed until the $(i-1)^{\text{th}}$ selection is made before considering the i^{th} selection. Thus, $k_{i-1} \le k_i$ for each $i \in \{2, \ldots, s\}$. $\square$

By the definition of the $Q_j(k), Q_j^o(k), \bar{Q}_j(k)$ probabilities, we know that they only depend on k, N, and the number of selections left before interviewing the current applicant, i.e. the subscript j. Thus, for two different models where the number of selections are s_1 and s_2 respectively (w.l.o.g. $s_1 < s_2$), with the same value of N, each of the thresholds k'_{s_1+1-j} for the model with s_1 selections and k''_{s_2+1-j} for the model with s_2 selections are the same for $1 \le j \le s_1$. In other words, an optimal strategy is right-hand based. This leads to the following result.

Corollary 2. *Let N be a fixed positive integer. There is a sequence of numbers $a_1, a_2, \ldots,$ such that when the number of selections $s \ge 1$ is fixed, then an optimal strategy is the $(a_s, a_{s-1}, \ldots, a_1)$-strategy. In other words, the $(s+1-i)^{th}$ threshold k_{s+1-i} (the i^{th} from the right) does not depend on the total number of selections allowed, i.e. the value of s, and always equals a_i, for $1 \le i \le s$.*

IV. Complete Problem Solution

Let $0 \le k_1 \le k_2 \le \ldots \le k_s \le N$. The $(k_1, k_2, \ldots, k_s)$-strategy is a strategy for which when making the i^{th} selection, where $1 \le i \le s$, we (1) wait until the $(i-1)^{\text{th}}$ selection is made (if $i = 1$ then there is no need to wait) and (2) reject the first k_i applicants and then accept the next left-to-right maximum. Let $0 \le r \le s$. We call a permutation π with at most N elements a $(k_1, k_2, \ldots, k_s)$-*r-choosable permutation* if we can make at most r selections when applying the $(k_1, k_2, \ldots, k_s)$-strategy on π.

Finally, let $0 \le k_1 \le k_2 \le \ldots \le k_s \le N$. We denote the number of $(k_1, k_2, \ldots, k_s)$-r-choosable permutations in S_N by

TABLE I

THE OPTIMAL THRESHOLDS (THRESHOLD RATIOS) AND SUCCESS PROBABILITIES.

	x_1	x_2	x_3	x_4	x_5	$\dots$
Thresholds	0.3678794412	0.2231301601	0.1410933807	0.0910176906	0.0594292419	$\dots$
Optimal probability	0.3678794412	0.5910096013	0.7321029820	0.8231206726	0.8825499146	$\dots$

$T_r(N, k_1, k_2, \dots, k_s)$. Furthermore, we denote the number of permutations in S_N such that the value N can be selected using the $(k_1, k_2, \dots, k_s)$-strategy by $W(N, k_1, k_2, \dots, k_s)$.

Note that the winning probability by using the $(k_1, k_2, \dots, k_s)$-strategy is $W(N, k_1, k_2, \dots, k_s)/N!$. We first obtain a recurrence relation on $T_{r-1}(m, k_1, \dots, k_r)$ and then use it to obtain a recurrence relation on $W(m, k_1, \dots, k_r)$.

Lemma 3. *Let $0 \le k_1 \le k_2 \le \dots \le k_s \le N$ be fixed, $1 \le r \le s$, and $m \ge k_r$. Then*

$$T_{r-1}(m, k_1, \dots, k_r) = (m-1)!\Big\{k_r + \sum_{i=k_r+1}^{m} \frac{T_{r-2}(i-1, k_1, \dots, k_{r-1})}{(i-1)!}\Big\},$$

while for $m \ge k_r + 1$ we have

$$W(m, k_1, \dots, k_r) = T_{r-1}(m-1, k_1, \dots, k_r) + (m-1) \cdot W(m-1, k_1, \dots, k_r).$$

By Corollary 2, we know that an optimal strategy is right hand based. Let $s \ge 1$ be fixed. By solving the recurrence relations found in Lemma 3, we obtain

$$\frac{W(N, a_s, \dots, a_1)}{N!} = \sum_{i=1}^{s} H_i, \quad \text{where}$$

$$H_s = \frac{a_s}{N} \cdot \Big(\sum_{i=a_s}^{a_{s-1}-1} \frac{1}{i} + \sum_{i_1=a_{s-1}+1}^{a_{s-2}-1} \frac{1}{i_1} \sum_{i_2=a_{s-1}}^{i_1-1} \frac{1}{i_2} + \sum_{i_1=a_{s-2}+1}^{a_{s-3}-1} \frac{1}{i_1} \sum_{i_2=a_{s-2}}^{i_1-1} \frac{1}{i_2} \sum_{i_3=a_{s-1}}^{i_2-1} \frac{1}{i_3} + \dots + \sum_{i_1=a_1+1}^{N-1} \frac{1}{i_1} \sum_{i_2=a_1}^{i_1-1} \frac{1}{i_2} \sum_{i_3=a_2}^{i_2-1} \frac{1}{i_3} \cdots \sum_{i_s=a_{s-1}}^{i_{s-1}-1} \frac{1}{i_s} \Big).$$

A. Asymptotics

It is also of interest to analyze the model when $N \to \infty$. In this case we need to normalize certain quantities. In particular, define $x_i = \lim_{N\to\infty} \frac{a_i}{N}$. For $0 \le x_s \le x_{s-1} \le \dots \le x_1 \le 1$, and $1 \le r \le s$, as $N \to \infty$ we have

$$H_r \to x_r \cdot \Big(\int_{x_r}^{x_{r-1}} \frac{1}{t}\, dt + \int_{x_{r-1}}^{x_{r-2}} \frac{1}{t_1} \int_{x_{r-1}}^{t_1} \frac{1}{t_2}\, dt_2\, dt_1 + \int_{x_{r-2}}^{x_{r-3}} \frac{1}{t_1} \int_{x_{r-2}}^{t_1} \frac{1}{t_2} \int_{x_{r-1}}^{t_2} \frac{1}{t_3}\, dt_3\, dt_2\, dt_1 + \dots + \int_{x_1}^{1} \frac{1}{t_1} \int_{x_1}^{t_1} \frac{1}{t_2} \int_{x_2}^{t_2} \cdots \int_{x_{r-1}}^{t_{r-1}} \frac{1}{t_r}\, dt_r\, dt_{r-1} \cdots dt_1 \Big) =: H'_r$$

$$\lim_{N\to\infty} \frac{W(N, a_s, \dots, a_1)}{N!} = \sum_{r=1}^{s} H'_r =: P.$$

For $1 \le i \le s$, the optimal probability is a function of $x_1, \dots, x_s$ and does not depend on N. By Corollary 2, we can first find x_1 for the case $s = 1$, then find x_2 for the case $s = 2$, ..., and so on up to x_s. An optimal strategy has to satisfy: (i) if $a_i = o(N)$, then $x_i = 0$; (ii) if $N - a_i = o(N)$, then $x_i = 1$; (iii) otherwise, $\lim_{N\to\infty} \frac{a_i}{N} = x_i \in (0, 1)$. Only (iii) is possible, i.e., each $x_i \in (0, 1)$, for $1 \le i \le s$. Indeed

$$\mathcal{I}_{r-1} = \int_{x_{r-1}}^{x_{r-2}} \frac{1}{t_1} \int_{x_{r-1}}^{t_1} \frac{1}{t_2}\, dt_2\, dt_1 + \dots + \int_{x_1}^{1} \frac{1}{t_1} \int_{x_1}^{t_1} \frac{1}{t_2} \int_{x_2}^{t_2} \cdots \int_{x_{r-1}}^{t_{r-1}} \frac{1}{t_r}\, dt_r\, dt_{r-1} \cdots dt_1.$$

We know that $\mathcal{I}_{r-1}$ is a constant when $x_1, \dots, x_{r-1}$ are given and the optimal value of x_r which realizes the maximum value of P equals $x_{r-1} \cdot e^{\mathcal{I}_{r-1}-1}$. Thus, we can compute x_{r-1} and $\mathcal{I}_{r-1}$ sequentially.

When N is large, by Corollary 2, an optimal strategy for the case of s selections is the $(x_s \cdot N, \dots, x_1 \cdot N)$-strategy: we reject the first $x_s \cdot N$ applicants and then select the first left-to-right maximum applicant thereafter; for the second selection, we wait until after the $(x_{s-1} \cdot N)^{\text{th}}$ position and then select the left-to-right maximum after the first selection made, ..., for the s^{th} selection, we wait until after the $(x_1 \cdot N)^{\text{th}}$ position and then select the left-to-right maximum after the $(s-1)^{\text{th}}$ selection is made. The results for $s \le 5$ are shown in Table I.

B. The expected stopping position

The expected position of stopping divided by N (ESR in short) for both the Dowry model and our query-base model are $0.7357N$ when $s = 1$. However, when s is large, the two models behave very differently. In the Dowry model, ESR approaches 1 as s becomes large. In the query-based model, ESR approaches 0.5 as $s \to \infty$. This follows since when s is sufficiently large, we have a success probability close to 1 and will stop at the position at which the value N appears if this is identified during the $s - 1$ queries (even for $s = 6$ we will have a probability > 0.9 of identifying the best applicant). Thus, ESR $\to 0.5$, the expected position of N. However, in the Dowry model, the probability of successfully identifying the best applicant during the first $s - 1$ selections $\to 1$ as s increases. Furthermore, since we have no information about whether the selected applicant is the best or not, we try our best to use all selections and thus have probabilities $\to 1$ of stopping at the end of the list. Thus, ESR $\to 1$ as $N \to \infty$ in the Dowry model.

The readers are referred to [16] for the full version.

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