Performance of CUSUM Tests for Detecting Changes in Continuous Time Processes

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Abstract — **We present a general methodology, for computing the worst mean detection delay of the CUSUM test for detecting changes in continuous time processes. The proposed method is applied to the problem of detecting changes in the constant drift of a Brownian motion and in the rate of a homogeneous Poisson process. Closed form expressions obtained for the latter case are completely novel.**

I. Introduction

The problem of sequentially detecting changes in the statistics of a random process has numerous applications in practice. Among the tests used for this problem the most popular one, is definitely the Cumulative Sum (CUSUM). Optimality of this scheme has been established, under Lorden's criterion, in discrete time for i.i.d. processes in [3]; and in continuous time for Brownian motion with constant drift in [2].

The performance of the CUSUM test is measured through the worst mean detection delay. The only case where a closed form expression exists, is for the detection of a change in the constant drift of a Brownian motion.

In this work we propose a general methodology for computing the performance of CUSUM tests for continuous time processes. Our method will be applied to the detection of a change in the drift of a Brownian motion, where we will rediscover the existing formulas; and the detection of a change in the rate of a Poisson process, where the corresponding expressions, to our knowledge, are completely novel.

II. CUSUM Test and Mean Detection Delay

Let $\{\xi_t\}_{t\geq 0}$ be a continuous time process. Let $\mathbb{P}_1(t)$ denote the probability measure when there is a change at 0 and $\mathbb{P}_0(t)$ when there is no change at all and let $u_t = \log(d\mathbb{P}_1(t)/d\mathbb{P}_0(t))$ be the corresponding log-likelihood ratio. Let finally $m_t = \min_{0\leq s\leq t} u_s$ be the running minimum of u_t. The CUSUM statistics y_t is then defined as $y_t = u_t - m_t \geq 0$.

The CUSUM test consists in issuing an alarm the first time y_t exceeds a constant threshold ν. Equivalently if T_c denotes the corresponding stopping time then we can define

$$T_C = \inf_t\{t : y_t \geq \nu\}.$$

The performance of T_c is measured through the mean detection delay $\mathbb{E}_1\{T_c\}$ and the mean delay between false alarms $\mathbb{E}_0\{T_c\}$.

In order to be able to estimate the two mean delays we propose the following methodology. A first step is to express the differential of a nonlinear transformation $f(y_t)$ of y_t as

$$df(y_t) = \mathcal{A}f(y_{t-})dt + \mathcal{B}f(y_{t-})dm_t + \mathcal{C}f(y_{t-})dw_t$$

where $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ are operators and w_t a martingale. Integrating the above equation and taking expectation we obtain

$$\mathbb{E}\{f(y_{T_C})\} - f(y_0) = \mathbb{E}\left\{\int_0^{T_C} \mathcal{A}f(y_{t-})dt + \mathcal{B}f(y_{t-})dm_t\right\}.$$

The second step consists in selecting the function $f(\cdot)$ so that $\mathcal{A}f(y) = -1$, $0 \leq y < \nu$; $\mathcal{B}f(y)|_{y=0} = 0$ and $f(y) = 0$, $y \geq \nu$. These conditions result in boundary value problems for the determination of $f(\cdot)$; either with (partial) differential equations or with differential-difference equations. Selecting $f(\cdot)$ as indicated above yields $\mathbb{E}\{T_C\} = f(y_0)$.

III. Applications

Applying our idea to a Brownian motion $d\xi_t = \mu_i dt + dw_t$, $i = 0, 1$ we obtain

$$\mathbb{E}_1\{T_C\} = \frac{2}{\mu^2}\{\nu + e^{-\nu} - 1\}, \quad \mathbb{E}_0\{T_C\} = \frac{2}{\mu^2}\{e^{\nu} - \nu - 1\},$$

where $\mu = \mu_1 - \mu_0$. The above formulas agree completely with the ones in the literature [1].

A notably more interesting situation occurs when ξ_t is a homogeneous Poisson processes with rates μ_i, $i = 0, 1$ before and after the change. Here we have to distinguish the two cases $\mu_1 > \mu_0$ and $\mu_1 < \mu_0$.

Case $\mu_1 > \mu_0$: Following our methodology we obtain

$$\begin{aligned}\mathbb{E}_i\{T_c\} &= \frac{1}{\mu_i}\sum_{n=0}^{[\frac{\nu}{\beta}]}\left\{1 - e^{\mu_i(\nu-n\beta)/\alpha}\sum_{k=0}^{n}\frac{\left(-\frac{\mu_i(\nu-n\beta)}{\alpha}\right)^k}{k!}\right\}\\ &\quad + \frac{p_i}{\mu_i}\sum_{n=0}^{[\frac{\nu}{\beta}]} e^{\mu_i(\nu-n\beta)/\alpha}\frac{\left(-\frac{\mu_i(\nu-n\beta)}{\alpha}\right)^n}{n!},\end{aligned}$$

where $[\cdot]$ denotes integer part; $\alpha = \mu_1 - \mu_0$; $\beta = \log(\mu_1/\mu_0)$; and $p_i = \frac{A_i}{A_i - B_i}$, with

$$\begin{aligned}A_i &= \sum_{n=0}^{[\frac{\nu}{\beta}]} e^{\mu_i(\nu-n\beta)/\alpha}\frac{\left(-\frac{\mu_i(\nu-n\beta)}{\alpha}\right)^n}{n!}\\ B_i &= \sum_{n=1}^{[\frac{\nu}{\beta}]} e^{\mu_i(\nu-n\beta)/\alpha}\frac{\left(-\frac{\mu_i(\nu-n\beta)}{\alpha}\right)^{n-1}}{(n-1)!}.\end{aligned}$$

Case $\mu_1 < \mu_0$: For this case, we obtain

$$\mathbb{E}_i\{T_C\} = \frac{1}{\mu_i}\sum_{n=0}^{[\frac{\nu}{|\beta|}]}\left\{e^{-\mu_i(\nu+n\beta)/\alpha}\left(\sum_{k=0}^{n}\frac{\left(\frac{\mu_i(\nu+n\beta)}{\alpha}\right)^k}{k!}\right) - 1\right\}.$$

Although not proven here (the paper is under preparation), it should be noted that the CUSUM test is optimum for the detection of a change in the rate of a homogeneous Poisson process.

References

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