CUSUM rules for detecting a regime change in the Brownian Motion model with multiple alternatives

Olympia Hadjiliadis*, George V. Moustakides†

May 2, 2004

Abstract. This work examines the problem of sequential change detection in the constant drift of a Brownian motion in the case of multiple alternatives. As a performance measure an extended Lorden's criterion is proposed. When the possible drifts, assumed after the change, have the same sign, the CUSUM rule designed to detect the smallest in absolute value drift, is proven to be the optimum. If the drifts have opposite signs then a specific 2-CUSUM rule is shown to be asymptotically optimal as the frequency of false alarms tends to infinity.

Key words. Change Detection, Quickest detection, CUSUM, Two-sided CUSUM.

1 Introduction and Mathematical formulation of the problem.

We begin by considering the observation process $\{\xi_t\}_{t>0}$ with the following dynamics:

$$d\xi_t = \begin{cases} dw_t & t \leq \theta \\ \mu_i \, dt + dw_t & t > \theta, \quad i = 1, 2, \end{cases}$$

where θ, the time of change, is assumed deterministic but unknown; μ_i, the possible drifts the process can change to, are assumed known, but the specific drift the process is changing to is assumed to be unknown. Our goal is to detect the change and *not to infer which of the changes occurred.*

The observation process $\{\xi_t\}_{t>0}$ generates the filtration $\{\mathcal{F}_t\}_{t>0}$ on the space $\Omega = C[0,\infty)$, as well as the families of probability measures

1. $\{\mathcal{P}^i_\theta\}, \ \theta \in [0,\infty)$, whenever the change is $\mu_i, \ i = 1, 2$.
2. $\mathcal{P}_\infty$, the Wiener measure, whenever no change takes place.

The objective is to detect the change as soon as possible while at the same time controlling the frequency of false alarms. This is achieved through the means of a stopping rule τ adapted to the filtration $\mathcal{F}_t$. One of the possible performance measures of the detection delay, suggested by Lorden in [5], considers the worst detection delay over all paths before the change and all possible change points θ. It is

$$J(\tau) = \sup_\theta \operatorname{ess\,sup} E_\theta \left[(\tau - \theta)^+ | \mathcal{F}_\theta \right], \tag{1}$$

giving rise to the following constrained stochastic optimization problem:

$$\inf_\tau J(\tau); \text{ subject to } E_\infty[\tau] \geq T. \tag{2}$$

In order to incorporate the different possibilities for the μ_i we extend Lorden's performance measure inspired by the idea of the worst detection delay regardless of the change (along the lines of [3]). It is

$$J_L(\tau) = \max_i \sup_\theta \operatorname{ess\,sup} E^i_\theta \left[(\tau - \theta)^+ | \mathcal{F}_\theta \right], \tag{3}$$

*Department of Statistics, Columbia University, New York, USA (oh66@columbia.edu).
†Department of Computer and Communication Engineering, University of Thessaly, Volos, Greece (moustaki@uth.gr).

which results in a corresponding optimization problem of the form:

$$\inf_{\tau} J_L(\tau);\ \text{subject to } E_\infty[\tau] \geq T. \tag{4}$$

It is easily seen, that in seeking solutions to the above problem, we can restrict our attention to stopping times that satisfy the false alarm constraint with equality. This is because, if $E_\infty[\tau] > T$, we can produce a stopping time that achieves the constraint with equality without increasing the detection delay, simply by randomizing between τ and the stopping time that is identically 0, as first advocated by [6]. To this effect, we introduce the following definition:

Definition 1 $\mathcal{K} = \{\tau \in \mathcal{F}_t;\ E_\infty[\tau] = T\}$.

The paper is organized as follows: In Section 2 the one-sided CUSUM stopping rule along with its optimal character is presented. Section 3 is devoted to the presentation of the 2-CUSUM stopping rules and certain families amongst them that display interesting properties. Finally, in Section 4, two asymptotic optimality results are presented as $T \to \infty$.

2 The one-sided CUSUM stopping time

The CUSUM statistic process and the corresponding one-sided CUSUM stopping time are defined as follows:

Definition 2 *Let $\lambda \in \mathcal{R}$ and $\nu \in \mathcal{R}_+$. Define the following processes:*

1. $u_t(\lambda) = \lambda\xi_t - \frac{1}{2}\lambda^2 t$; $m_t(\lambda) = \inf_{0\leq s\leq t} u_s(\lambda)$.
2. $y_t(\lambda) = u_t(\lambda) - m_t(\lambda) \geq 0$, *which is the CUSUM statistic process.*
3. $\tau_c(\lambda,\nu) = \inf\{t \geq 0; y_t(\lambda) \geq \nu\}$, *which is the CUSUM stopping time.*

We are now in a position to examine two very important properties of the one-sided CUSUM stopping time. The first is seen in the following lemma:

Lemma 1 *Fix $\theta \in [0,\infty)$. Let $t \geq \theta$ and consider the CUSUM process when starting at time θ, $y_{t,\theta} = u_t - u_\theta - \inf_{\theta\leq s\leq t}(u_t - u_\theta)$. We have that $y_t \geq y_{t,\theta}$ with equality if $y_\theta = 0$.*

Proof: The proof comes as a result of noticing that $y_t = y_{t,\theta} + \Big(\inf_{\theta\leq s\leq t}(u_s - u_\theta) + y_\theta\Big)^+ \geq y_{t,\theta}$ and that $\inf_{\theta\leq s\leq t}(u_s - u_\theta) \leq 0$. $\diamond$

By its definition it is clear that $y_{t,\theta}$ depends only on information received *after* time θ. Thus, we conclude that all contribution of the observation process $\{\xi_t\}$ before time θ, is summarized in y_θ. Lemma 1, therefore suggests that the worst detection delay *before* θ occurs whenever $y_\theta = 0$. In other words,

$$\operatorname{ess\,sup} E_\theta\left[(\tau_c(\lambda,\nu) - \theta)^+ | \mathcal{F}_\theta\right] = E_\theta\left[(\tau_c(\lambda,\nu) - \theta)^+ | y_\theta = 0\right] = E_0\left[\tau_c(\lambda,\nu)\right]. \tag{5}$$

Equ. (5) states that the CUSUM stopping time is an equalizer rule over θ, in the sense that its performance does not depend on the value of this parameter.

The second property of the one-sided CUSUM comes as a result of noticing that m_t is nonincreasing and that when it changes (decreases) we necessarily have $m_t = u_t$. In other words, when m_t changes, y_t attains its smallest value, that is 0. When this happens we will say that the CUSUM statistic process *restarts.*

Lemma 2 *Suppose a CUSUM stopping rule is based on the CUSUM statistic with drift parameter $\lambda \in \mathcal{R}$ and has threshold $\nu \in \mathcal{R}_+$. Then, the detection delay when the observation process ξ_t has drift $\mu \in \mathcal{R}$ is given by $E[\tau_c(\lambda,\nu)] = (2/\lambda^2)g(\nu,\rho)$, where*

$$g(\nu,\rho) = \frac{e^{-\rho\nu} + \rho\nu - 1}{\rho^2} \quad \text{and} \quad \rho = 2\frac{\mu}{\lambda} - 1.$$

Proof: The proof follows from the second property of the one sided CUSUM combined with standard stochastic calculus results (see [4]). For more details refer to [7]. $\diamond$

Notice that for $\alpha \neq 0$ we have $\frac{1}{\alpha^2} g(\nu,\rho) = g(\frac{\nu}{|\alpha|}, \rho|\alpha|)$. This suggests the following alternative expression for the delay function

$$E[\tau_c(\lambda,\nu)] = 2g\left(\frac{\nu}{|\lambda|}, \mathrm{sign}(\lambda)(2\mu-\lambda)\right). \tag{6}$$

In [2] and [8] it is shown that when there is only one possible alternative for the drift μ, the CUSUM stopping rule $\tau_c(\mu,\nu)$, with ν satisfying $\frac{2}{\mu^2} g(\nu,-1) = T$, solves the optimization problem defined in (2). It is also interesting to note that in [7], after a proper modification of Lorden's criterion that replaces expected delays with Kullback-Leibler divergences, the optimality of the CUSUM can be extended to cover detection of general changes in Itô processes.

When the sign of the alternative drifts is the same, with the help of the following lemma we can show that the one-sided CUSUM stopping rule that detects the smallest in absolute value drift is the optimal solution of the problem in (4).

Lemma 3 *For every path of the Brownian motion w_t, the process $y_t(\lambda)$ is an increasing (decreasing) function of the drift of the observation process ξ_t when $\lambda > 0$ ($\lambda < 0$).*

Proof: Consider two possible drift values μ_1, μ_2 with $\mu_1 < \mu_2$. We define the following two observation processes $\xi_t(\mu_i) = \mu_i(t-\theta)^+ + w_t,\ i = 1, 2$, that lead to the corresponding CUSUM processes

$$\begin{aligned} u_t(\lambda,\mu_i) &= \lambda\xi_t(\mu_i) - \frac{1}{2}\lambda^2 t = \lambda\{w_t + \mu_i(t-\theta)^+\} - \frac{1}{2}\lambda^2 t; \quad m_t(\lambda,\mu_i) = \inf_{0\le s\le t} u_s(\lambda,\mu_i) \\ y_t(\lambda,\mu_i) &= u_t(\lambda,\mu_i) - m_t(\lambda,\mu_i). \end{aligned}$$

Consider the difference $y_t(\lambda,\mu_2) - y_t(\lambda,\mu_1) = \delta(t-\theta)^+ - m_t(\lambda,\mu_2) + m_t(\lambda,\mu_1)$ where $\delta = \lambda(\mu_2-\mu_1)$. Notice now that $\lambda > 0$ implies $\delta > 0$ and we can write

$$u_s(\lambda,\mu_2) = u_s(\lambda,\mu_1) + \delta(s-\theta)^+ \le u_s(\lambda,\mu_1) + \delta(t-\theta)^+.$$

Taking the infimum over $0 \le s \le t$ we get $m_t(\lambda,\mu_2) \le m_t(\lambda,\mu_1) + \delta(t-\theta)^+$ from which, by rearranging terms, we get that $y_t(\lambda,\mu_2) \ge y_t(\lambda,\mu_1)$. The case $\lambda < 0$ can be shown similarly. $\diamond$

From Lemma 3 it also follows that $\mu_1 \le \mu_2$ implies $E^1[\tau_c(\lambda,\nu)] \ge E^2[\tau_c(\lambda,\nu)]$ when $\lambda > 0$ and the opposite when $\lambda < 0$. As a direct consequence of this fact comes our first optimality result concerning drifts with the same sign.

Theorem 1 *Let $0 < \mu_1 \le \mu_2$ or $\mu_2 \le \mu_1 < 0$, then the one-sided CUSUM stopping time $\tau_c(\mu_1,\nu_1)$ with ν_1 satisfying $\frac{2}{\mu_1^2} g(\nu_1,-1) = T$ solves the optimization problem defined in (4).*

Proof: The proof is straightforward. Since ν_1 was selected so that $\tau_c(\mu_1,\nu_1)$ satisfies the false alarm constraint, we have $\tau_c(\mu_1,\nu_1) \in \mathcal{K}$. Then, $\forall\ \ \tau \in \mathcal{K}$ we have

$$\begin{aligned} J_L(\tau) &= \max_i \sup_\theta \operatorname{ess\,sup} E_\theta^i\left[(\tau-\theta)^+|\mathcal{F}_\theta\right] \ge \sup_\theta \operatorname{ess\,sup} E_\theta^1\left[(\tau-\theta)^+|\mathcal{F}_\theta\right] \\ &\ge E_0^1[\tau_c(\mu_1,\nu_1)] = \max_i E_0^i[\tau_c(\mu_1,\nu_1)] = J_L(\tau_c(\mu_1,\nu_1)) = \frac{2}{\mu_1^2} g(\nu_1, 1). \end{aligned}$$

The last inequality comes from the optimality of the one-sided CUSUM stopping rule and the last three equalities are due to Lemmas 3, 2 and the definition of $J_L(\tau)$ in (3) $\diamond$

It is worth pointing out that if we had n alternative drifts (instead of two) of the form $0 < \mu_1 \le \mu_2 \le \cdots \le \mu_n$ or $0 > \mu_1 \ge \mu_2 \ge \cdots \ge \mu_n$ and we used the extended Lorden's criterion in (3), the optimality of $\tau_c(\mu_1,\nu_1)$, presented in Theorem 1, would still be valid. Our result should be compared to [3] (which refers to discrete time and the exponential family), where for the same type of changes only asymptotically optimum schemes are offered.

We also have the following corollary of Lemma 3:

Corollary 1 *Let $0 < |\mu_1| \le |\mu_2|$ and define $\eta_i,\ i = 1, 2,$ so that $\frac{2}{\mu_i^2} g(\eta_i, -1) = T > 0$. Then we have*

$$\frac{1}{\mu_1^2} g(\eta_1, 1) \ge \frac{1}{\mu_2^2} g(\eta_2, 1). \tag{7}$$

Proof: Since the result is independent of the sign of the two drifts, without loss of generality we may assume $0 < \mu_1 \le \mu_2$. Consider the two CUSUM rules $\tau_c(\mu_i, \eta_i),\ i = 1, 2$. Because the two thresholds η_i were selected to satisfy the false alarm constraint, using Lemma 1, Lemma 3 and the optimality of the one-sided CUSUM stopping time, the following inequalities hold $\forall\ \tau \in \mathcal{K}$:

$$\begin{aligned} \frac{2}{\mu_1^2} g(\eta_1, 1) &= E_0^1[\tau_c(\mu_1, \eta_1)] \ge E_0^2[\tau_c(\mu_1, \eta_1)] = \sup_\theta \operatorname{ess\,sup} E_\theta^2 \left[(\tau_c(\mu_1, \eta_1) - \theta)^+ | \mathcal{F}_\theta \right] \\ &\ge \inf_\tau \sup_\theta \operatorname{ess\,sup} E_\theta^2 \left[(\tau - \theta)^+ | \mathcal{F}_\theta \right] = E_0^2[\tau_c(\mu_2, \eta_2)] = \frac{2}{\mu_2^2} g(\eta_2, 1). \quad \diamond \end{aligned}$$

3 Different drift signs and the 2-CUSUM stopping time

Let us now consider the case $\mu_2 < 0 < \mu_1$. The very interesting problem of knowing the amplitude of the drift but not the sign falls into this setting. What has traditionally been done in the literature, dating as far back as Barnard in [1], is to use the minimum of the stopping rules $\tau_c(\mu_1, \nu_1)$ and $\tau_c(\mu_2, \nu_2)$ each tuned to detect the respective changes μ_1 and μ_2. To this effect, we introduce the following 2-CUSUM stopping rule:

Definition 3 *Let $\lambda_2 < 0 < \lambda_1$. The 2-CUSUM stopping time $\tau_{2c}(\lambda_1, \lambda_2, \nu_1, \nu_2)$ is defined as follows: $\tau_{2c}(\lambda_1, \lambda_2, \nu_1, \nu_2) = \tau_c(\lambda_1, \nu_1) \wedge \tau_c(\lambda_2, \nu_2)$.*

In the sequel, all 2-CUSUM rules will be denoted by τ_{2c} unless it is necessary to give emphasis to their four parameters. From Lemma 1, it follows that:

$$J_L(\tau_{2c}) = \max_i \sup_\theta \operatorname{ess\,sup} E_\theta^i \left[(\tau_{2c} - \theta)^+ | \mathcal{F}_\theta \right] = \max_i E_0^i \left[\tau_{2c} \right]. \tag{8}$$

As we have seen the 2-CUSUM stopping rule is characterized by the four parameters, $\lambda_1, \lambda_2, \nu_1$ and ν_2. Since our intention is to propose a specific rule as the "preferable" we will gradually impose additional constraints (apart from the false alarm constraint) on our 2-CUSUM structure in order to arrive to a unique stopping rule. Once our rule is specified we will support its selection by demonstrating that it enjoys a strong asymptotic optimality property.

3.1 A special class of 2-CUSUM rules

First we shed our attention to a specific class of 2-CUSUM stopping rules that allow for the exact computation of their performance.

Definition 4 *Define $\mathcal{G} = \{\tau_{2c}(\lambda_1, \lambda_2, \nu_1, \nu_2); \nu_1 = |\lambda_1|\nu$ and $\nu_2 = |\lambda_2|\nu\}$.*

Lemma 4 *Let $\tau_{2c} \in \mathcal{G}$ then, when τ_{2c} stops, one of its CUSUM statistic processes hits its corresponding threshold while the other necessarily restarts.*

Proof: Although the proof given in [9, Page 28] for discrete time and the exponential family, applies here as well, we prefer to give an alternative (hopefully easier) proof. Consider the sum

$Y_t = |\lambda_2| y_t(\lambda_1) + |\lambda_1| y_t(\lambda_2) = -\frac{1}{2}(|\lambda_2|\lambda_1^2 + |\lambda_1|\lambda_2^2)t - |\lambda_2| m_t(\lambda_1) - |\lambda_1| m_t(\lambda_2)$.

We notice that when neither of the two CUSUM processes $y_t(\lambda_i),\ i = 1, 2,$ restarts, the corresponding processes $m_t(\lambda_i)\ \forall\ i = 1, 2$ stay constant, which implies that Y_t *decreases linearly* in time. From this we conclude that Y_t can increase only when one of the two CUSUM processes $y_t(\lambda_i)$ restarts. We obviously cannot have both CUSUM processes restarting, since that would imply $Y_t = 0$. Therefore, the 2-CUSUM stops when one of the two CUSUM statistic processes hits its corresponding threshold. This occurs when Y_t attains the level $|\lambda_1\lambda_2|\nu > 0$ *for the first time*. Since Y_t attains a new level, it has to be during an increase of Y_t, which can only happen when one of the CUSUMs restarts. $\diamond$

Lemma 5 *Let $\tau_{2c} = \tau_1 \wedge \tau_2$ with $\tau_{2c} \in \mathcal{G}$ and τ_1, τ_2 the corresponding one-sided CUSUM branches. Then the expected delay of the 2-CUSUM stopping time τ_{2c} is equal to:*

$$(E[\tau_{2c}])^{-1} = (E[\tau_1])^{-1} + (E[\tau_2])^{-1}. \tag{9}$$

Proof: The proof basically repeats the one presented in [9, Page 28] for the discrete time case. $\diamond$

3.2 2-CUSUM equalizer rules

It is well known that min-max problems, such as (4), are solved by *equalizer rules*. Thus, we further restrict ourselves among the class of equalizer rules.

Definition 5 *Define $\mathcal{D} = \{\tau_{2c} \in \mathcal{G}; E_0^1[\tau_{2c}] = E_0^2[\tau_{2c}]\}$.*

By the definition of the class of equalizer rules it follows that $\mathcal{D} \subset \mathcal{G}$. Using Eqs. (6), (9) we can see that in order to have $\tau_{2c} \in \mathcal{D}$ we need $\lambda_1 + \lambda_2 = 2(\mu_1 + \mu_2)$. We now proceed to select the parameter λ_1 so that the corresponding detection delay is asymptotically (as $T \to \infty$) minimized.

Theorem 2 *Let $\mu_2 < 0 < \mu_1$ with $|\mu_1| \le |\mu_2|$. Consider all 2-CUSUM stopping times $\tau_{2c} \in \mathcal{K} \cap \mathcal{D}$. Then among all such stopping rules the one with $\lambda_1 = \mu_1$, $\lambda_2 = 2\mu_2 + \mu_1$ is asymptotically optimal as $T \to \infty$.*

Proof: Since $\mu_1 + \mu_2 \le 0$ and $\tau_{2c} \in \mathcal{D}$, for any $\lambda_1 > 0$, we get $|\lambda_1| \le |\lambda_2|$. From the false alarm constraint we get $\lambda_1\nu = \log T(1 + o(1))$. Using Eqs. (6), (9) and substituting the expression for λ_2 in terms of λ_1, μ_1 and μ_2, which ensures that $\tau_{2c} \in \mathcal{D}$ we get:

$$E_0^i[\tau_{2c}] = \left(\frac{1}{2g(\nu, 2\mu_1 - \lambda_1)} + \frac{1}{2g(\nu, 2\mu_2 - \lambda_1)}\right)^{-1} = \begin{cases} \frac{2\nu}{2\mu_1 - \lambda_1}(1 + o(1)) & \text{for } 2\mu_1 > \lambda_1 \ge 0 \\ \nu^2(1 + o(1)) & \text{for } 2\mu_1 = \lambda_1 \\ \frac{2e^{\nu|2\mu_1 - \lambda_1|}}{(2\mu_1 - \lambda_1)^2}(1 + o(1)) & \text{for } 2\mu_1 < \lambda_1. \end{cases} \tag{10}$$

From (10) it is clear that it is sufficient to limit ourselves to the case $0 \le \lambda_1 < 2\mu_1$, since for $\lambda_1 \ge 2\mu_1$ the detection delay increases significantly faster as ν increases. For $0 \le \lambda_1 < 2\mu_1$, the detection delay, after substituting ν from the false alarm constraint, can be written as $\frac{2 \log T}{\lambda_1(2\mu_1 - \lambda_1)}(1 + o(1))$, which is clearly minimized, asymptotically, for $\lambda_1 = \mu_1$. Since $\tau_{2c} \in \mathcal{D}$, we also get $\lambda_2 = 2\mu_2 + \mu_1$. $\diamond$

4 Asymptotic optimality in opposite sign drifts

For the specific 2-CUSUM rule introduced at the end of the previous section, we are going to demonstrate two asymptotic optimality results by means of an upper and a lower bound on the performance of the unknown optimal stopping rule. We will show that the *difference* in performance between the unknown optimum rule and the proposed 2-CUSUM rule is bounded above by a constant (equal in absolute value drifts) or tends to 0 (unequal in absolute value drifts) as $T \to \infty$. This should be compared to most existing asymptotic optimality results (see [10]) where it is shown that the *ratio* between the performance of the optimum and the proposed scheme tends to unity (first order optimality). Our form of asymptotic optimality is clearly stronger since it implies first order optimality, while the opposite is not necessarily true.

Let τ_{2c} denote the specific 2-CUSUM rule proposed in the previous section with the threshold ν selected so that the false alarm constraint is satisfied with equality. Since τ_{2c} constitutes a possible choice in the class $\mathcal{K} \cap \mathcal{D}$, we have that $\forall\ \tau \in \mathcal{K}$

$$E_0^1[\tau_{2c}] = E_0^2[\tau_{2c}] = J_L(\tau_{2c}) \ge \inf_\tau J_L(\tau) \ge \max_i \inf_\tau \sup_\theta \operatorname{ess\,sup} E_\theta^i[(\tau - \theta)^+ | \mathcal{F}_\theta] = \max_i \frac{2}{\mu_i^2} g(\eta_i, 1) \tag{11}$$

where the two thresholds η_i, $i = 1, 2$, are selected to satisfy the false alarm constraint $\frac{2}{\mu_i^2} g(\eta_i, -1) = T$. The asymptotic results that follow examine the way the two bounds approach each other, which also determine the rate with which the 2-CUSUM approaches the optimal solution.

4.1 The case of equal in absolute value drifts

We first consider the special case $\mu_1 = -\mu_2 = \mu$. Here our parameter selection takes the form $\lambda_1 = \mu_1 = \mu$ and $\lambda_2 = 2\mu_2 + \mu_1 = \mu_2 = -\mu$ which coincides with the 2-CUSUM scheme proposed in the literature.

Theorem 3 *The difference in the performance between the proposed 2-CUSUM stopping rule and the optimal stopping rule is bounded above by a quantity that tends to the constant* $\frac{2\log 2}{\mu^2}$*, as the false alarm constraint* $T \to \infty$.

Proof: Solving for ν from the false alarm constraint $E_\infty[\tau_{2c}] = g(\nu, -\mu) = T$ we obtain $\mu\nu = \log T + \log \frac{\mu^2}{2} + \log 2 + o(1)$. On the other hand, we can write the upper bound in Equ.(11) as $J_L(\tau_{2c}) = \frac{2}{\mu^2}\{\mu\nu + e^{-\mu\nu} - 1\}\{1 + O(\mu\nu e^{-3\mu\nu})\}$. Substituting the estimate for ν we get $J_L(\tau_{2c}) = \frac{2}{\mu^2}\left\{\log T + \log \frac{\mu^2}{2} - 1 + \log 2 + o(1)\right\}$. Similarly, for the lower bound we have that the threshold η as a function of T becomes $\eta = \log T + \log \frac{\mu^2}{2} + o(1)$, which follows from the false alarm constraint $\frac{2}{\mu^2} g(\eta, -1) = T$. As a result of substituting the above expression for η in $\frac{2}{\mu^2} g(\eta, 1)$, that is the lower bound in Equ. (11), we get $\frac{2}{\mu^2}\{\log T + \log \frac{\mu^2}{2} - 1 + o(1)\}$ for the lower bound. Since the difference between the upper and the lower bound, bounds the difference $J_L(\tau_{2c}) - \inf_\tau J_L(\tau)$, we conclude that

$$0 \le J_L(\tau_{2c}) - \inf_\tau J_L(\tau) \le \frac{2}{\mu^2}\{\log 2 + o(1)\},$$

from which the result follows by letting $T \to \infty$. $\diamond$

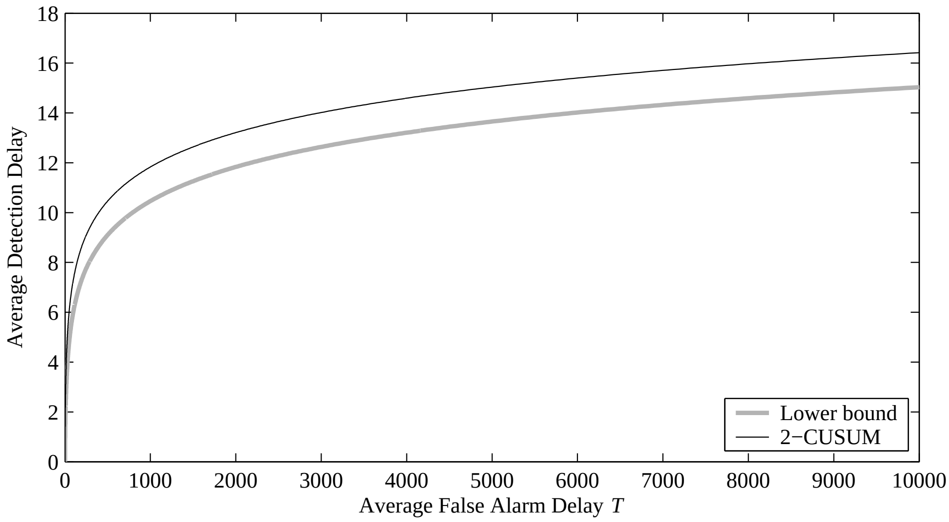

Figure 1: Typical form of the upper and lower bounds of the performance of the optimum stopping rule for the case $\mu_1 = -\mu_2 = 1$.

Figure 1 depicts the upper and lower bound as a function of the false alarm constraint T for the case $\mu_1 = -\mu_2 = 1$. Notice the difference of the two bounds is increasing with T. The constant proposed by Theorem 3 corresponds to a *worst case* performance attained only as $T \to \infty$.

4.2 The case of different in absolute value drifts

Theorem 4 *The difference in the performance between the proposed 2-CUSUM stopping rule and the optimal stopping rule is bounded and tends to 0, as the false alarm constraint $T \to \infty$.*

Proof: We will only examine the case $|\mu_1| < |\mu_2|$. From Corollary 1 and Equ. (7) it follows that the maximum in the lower bound in (11) is achieved for μ_1. Hence, as in Theorem 3, we get $\frac{2}{\mu_1^2}\{\log T + \log\frac{\mu_1^2}{2} - 1 + o(1)\}$ for the lower bound.

The upper bound is the detection delay of the proposed 2-CUSUM stopping time τ_{2c} in Theorem 2. Using Eqs (6) and (9), with $\lambda_1 = \mu_1$, $\lambda_2 = 2\mu_2 + \mu_1$, we have

$$J_L(\tau_{2c}) = E_0^i[\tau_{2c}] \;=\; \frac{2}{\mu_1^2}\{e^{-\mu_1\nu} + \mu_1\nu - 1\}\{1 + O(\mu_1\nu e^{(2\mu_2-\mu_1)\nu})\}, \tag{12}$$

where ν is selected to satisfy the false alarm constraint, which takes the form

$$E_\infty[\tau_{2c}] = \left(\frac{1}{2g(\nu,-\mu_1)} + \frac{1}{2g(\nu,2\mu_2+\mu_1)}\right)^{-1} = T.$$

From it we get the estimate

$$\mu_1\nu = \log T + \log\frac{\mu_1^2}{2} + o(1).$$

This, when substituted in (12), produces $J_L(\tau_{2c}) = E_0^i[\tau_{2c}] = \frac{2}{\mu_1^2}\left\{\log T + \log\frac{\mu_1^2}{2} - 1 + o(1)\right\}$.

Subtracting now the lower bound expression from the upper bound expression above we obtain

$$0 \le J_L(\tau_{2c}) - \inf_\tau J_L(\tau) \le o(1),$$

which tends to 0 as $T \to \infty$. $\diamond$

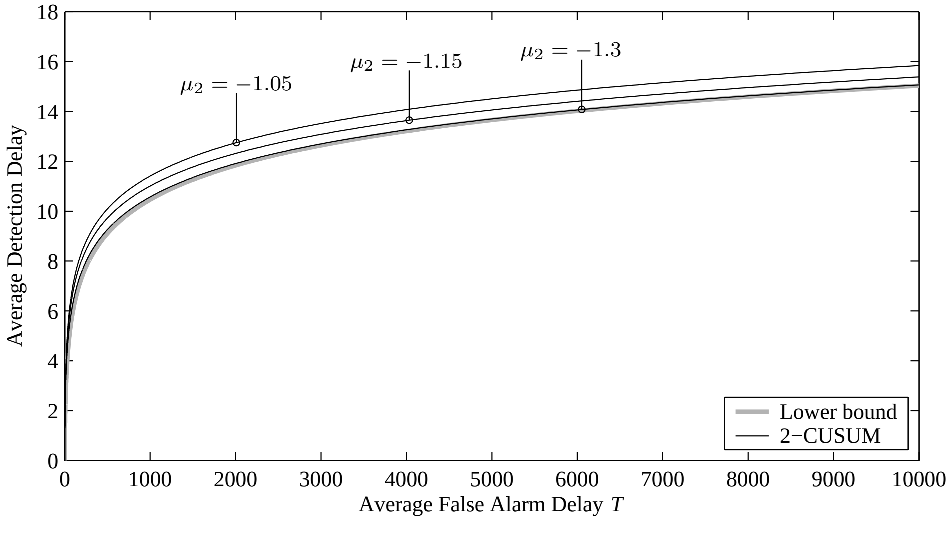

Figure 2: Typical form of the upper and lower bounds of the performance of the optimal stopping rule for the case $\mu_2 < 0 < \mu_1$, with $\mu_1 = 1$ and $\mu_2 = -1.05, -1.15, -1.3$.

In Figure 2 we present the two bounds for $\mu_1 = 1$ and $\mu_2 = -1.05, -1.15, -1.3$. We recall that the upper bound is the detection delay of the 2-CUSUM rule $\tau_{2c} \in \mathcal{G} \cap \mathcal{K}$ with parameters $\lambda_1 = \mu_1$ and $\lambda_2 = 2\mu_2 + \mu_1$. We can see that the difference between the two curves is tending to zero as the false alarm tends to infinity, thus corroborating Theorem 4. What is more interesting, however, is the fact that the two curves rapidly approach each other, *uniformly* over T, as the ratio $|\mu_2|/|\mu_1|$ of the two drifts increases. As we can see, in the case $\mu_1 = 1,\ \mu_2 = -1.3$ the two bounds become almost indistinguishable. This suggests that the proposed 2-CUSUM rule can be (extremely) close to the unknown optimal rule, not only asymptotically, as proposed by Theorem 4, but also uniformly over all false alarm values.

It is also worth noting that the difference in the performance of the optimal rule and any 2-CUSUM rule in $\mathcal{G}$ with parameters $\lambda_1 = \mu_1$ and $\lambda_2 \in (-\mu_1\ 2\mu_2 + \mu_1]$ (one such possibility is the selection proposed in the literature $\lambda_1 = \mu_1,\ \lambda_2 = \mu_2$) also tends to 0 as $T \to \infty$. Therefore, asymptotically optimal solutions allow for many different choices. It is, however, our selection that leads to an equalizer rule.

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