Both $P_F(\tau, i)$ and $P_D(\tau, i)$ are quasi-convex. Moreover, they are both strictly monotonically decreasing functions w.r.t. τ. So for every admissible fusion rule F_i, define τ_i to be the smallest τ such that $P_F(\tau, i) \leq \alpha_0$. Due to the monotonically decreasing property, this τ_i must satisfy $P_F(\tau_i, i) = \alpha_0$ and the corresponding $P_D(\tau_i, i)$ is the maximized detection probability for the admissible fusion rule F_i. Define

$$i^* = \arg\max_i P_D(\tau_i, i)$$

then F_{i^*} is the optimal fusion rule and the maximum detection probability is $P_D(\tau_{i^*}, i^*)$. This method also applies to the case with non-Gaussian noise distributions.

VI. Conclusion

In this correspondence, we considered distributed detection of $s \in \{-m, m\}$, where the ith of n local sensors observes $x_i = s + z_i$ with i.i.d. additive noise z_i. The ith sensor makes a binary decision u_i based on a threshold τ. A fusion center uses these decisions to produce the global decision using a fusion rule F.

When all admissible rules have the probability of error as a quasi-convex function of τ, the problem decomposes into a series of n quasi-convex optimization problems that may be solved using well-known techniques. We showed this quasi-convexity property for generalized Gaussian noise. For some non-Gaussian noise distributions we showed this quasi-convexity property when the hypotheses have equal *a priori* probability.

We also used the quasi-convexity perspective to provide solution techniques for Bayes risk and Neyman–Pearson formulations of the sensor data fusion problem.

Applying our solution technique to binary sensors in Gaussian noise reveals that the number of binary sensors needed for every SNR to achieve error probability of 10^{-5} is fewer than twice the number of infinite-precision sensors required. So the binary sensor can be a better choice from a practical or economic point of view.

Zhang *et al.* [14] generalize these results by showing quasi-convexity in the likelihood ratio function for any distribution on the i.i.d. observations x_i.

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On the Relative Error Probabilities of Linear Multiuser Detectors

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***Abstract*—The relative error-probability performance of three linear multiuser detectors—the minimum mean-square error detector, the decorrelator, and the conventional matched filter (MF) detector—is investigated under nonorthogonal signaling and additive white Gaussian noise conditions. It is shown that, contrary to the general belief, the minimum mean-square-error (MMSE) detector does not uniformly outperform the other two detectors. In fact, even for the two-user case, one can find counterexamples where the matched filter is significantly better.**

***Index Terms*—Code division multiple access (CDMA), multiuser detection.**

I. Introduction

Linear multiuser detection schemes have attracted considerable attention lately due to their simplicity, low complexity (as compared to optimum detection schemes), and satisfactory performance which, although not optimum in a minimum-error-rate sense, can nevertheless satisfy a number of alternative asymptotic optimization criteria such as high efficiency or near–far resistance [8, pp. 195–202].

The matched-filter (MF) detector is, of course, the simplest linear detector. Since this detector neglects the presence of interfering users,

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its performance can be extremely poor in the presence of severe multiple-access interference (MAI). Two key linear multiuser detectors are known to combat MAI very efficiently. Specifically, the *decorrelating* detector (or *decorrelator*) defeats MAI by selecting a proper linear filter to eliminate it [4], [8], while the *minimum mean-square-error* (MMSE) detector achieves robustness against MAI by selecting the linear filter that minimizes the mean-square value of the output MAI plus noise [2], [5], [7]–[9].

The MMSE detector has been the center of recent attention due to its noticeable feature of being practically implementable through blind adaptive schemes; that is, through schemes that use only the signature waveform of one user of interest and do not require training sequences or knowledge of signatures of interferers [2]. In [6] one can find a detailed analysis of this detector's performance under various conditions related to multiuser applications, along with efficient approximations of the corresponding error rates.

Several analytical and numerical results have suggested the conjecture (stated in [6]) that the MMSE detector outperforms the decorrelator for any combination of signal and noise powers. Furthermore, a similar (if not stronger) feeling seems to be shared when the MMSE detector is compared against the conventional MF detector. It is the aim of this work to prove that both of these conjectures are in fact false. Specifically, for the comparison of the MMSE detector versus the decorrelator we show that for the two-user case and for large enough cross correlation of the signature signals, it is possible to find noise and signal powers for which the MMSE detector is inferior to the decorrelator. We also show that such a case cannot appear under perfect power control conditions. For the comparison of the MF against the MMSE and the decorrelating detector we show that, for the general K-user case with essentially arbitrary cross correlations, the MF detector outperforms the other two detectors provided that the power of the user of interest is sufficiently large. Examples where the MF is better than the other two detectors are also presented for the perfect power control case.

Let us now define the problem of interest in some more detail. Consider a K-user binary communication system with corresponding normalized modulation waveforms $s_1, \ldots, s_K$, and signaling antipodally through an additive white Gaussian noise channel. If we limit ourselves to the synchronous signal case then the length-K vector y whose lth component is the output of a filter matched to s_l is a sufficient statistic for the problem of detecting the transmitted symbols. The vector y can be written as [8, p. 56]

$$y = RA\tilde{b} + \sigma n \tag{1}$$

where R denotes the normalized crosscorrelation matrix of the signals $s_1, \ldots, s_K$; A is the diagonal matrix $\operatorname{diag}\{A_1, \ldots, A_K\}$ with A_l the received amplitude of user l; $\tilde{b} = [b_1, \ldots, b_K]^t$ is a vector whose lth component is the symbol $b_l \in \{\pm 1\}$ of user l; n is a $\mathcal{N}(0, R)$ normal random vector independent of $\tilde{b}$; and, finally, σ^2 is the intensity of the additive channel noise.

As noted above, we wish to examine the relative performance of the MMSE detector, the decorrelator, and the MF detector. Without loss of generality throughout our analysis we will consider User 1 to be the user of interest. The corresponding error probabilities are then given by the following formulas [8, pp. 113, 249, 300]:

$$P_{\mathrm{MMSE}} = 2^{1-K} \sum_{b_i=\pm 1} Q\left(\frac{e_1^t (R + X^{-2})^{-1} RXb}{\sqrt{z(R, X)}}\right) \tag{2}$$

$$P_D = Q\left(\frac{x_1}{\sqrt{e_1^t R^{-1} e_1}}\right) \tag{3}$$

$$P_{MF} = 2^{1-K} \sum_{b_i=\pm 1} Q(e_1^t RXb) \tag{4}$$

where

$$z(R, X) = e_1^t (R + X^{-2})^{-1} R (R + X^{-2})^{-1} e_1$$
$$X = \operatorname{diag}\{x_1, \ldots, x_K\}$$

$x_l = A_l/\sigma$, $b = [1\ b_2 \ldots\ b_K]^t$, $e_1 = [1\ 0 \ldots 0]^t$, and $Q(\cdot)$ denotes the complementary unit cumulative Gaussian distribution. Finally, we should note that the sums in (2) and (4) are taken over all possible combinations of b_i, $i = 2, \ldots, K$. In the following sections, we compare these three expressions under various signaling conditions.

II. The Matched Filter versus the MMSE and Decorrelating Detectors

In this section, our goal is to show that the MF outperforms both the MMSE and decorrelating detectors provided the user of interest is sufficiently strong in power. In particular, we prove the following result.

Proposition 1: Fix $R, x_2, \ldots, x_K$, and assume that R is positive definite. If at least one interfering user has a signature waveform that is nonorthogonal to the signature waveform of User 1, then there exists sufficiently large x_1 for which the MF detector outperforms both the MMSE detector and the decorrelator.

Proof: The proof is presented in the Appendix. □

For the case where the signatures of the interfering users are orthogonal to the signature of User 1 we know that all three schemes have the same performance $P_{\mathrm{MMSE}} = P_D = P_{\mathrm{MF}} = Q(x_1)$, which is, of course, the single-user performance.

From Proposition 1 it is not clear what the size of x_1 must be in order for the proposition to hold. It turns out, at least for the two-user case, that even for moderate values of x_1 the conclusion of the proposition can hold. If ρ denotes the cross correlation of the normalized signature waveforms of the two users, then Fig. 1 depicts, as a function of ρ, the pairs (x_1, x_2) for which the detectors have the same performance. Specifically, Fig. 1(a) shows these pairs for the MF and MMSE detector, while Fig. 1(b) shows the corresponding pairs for the MF and the decorrelator. All points were obtained numerically. We also mention that in Fig. 1(a) points lying to the right of each curve correspond to the case where the MF outperforms the MMSE detector, while in Fig. 1(b) points below each curve correspond to combinations where MF is better than the decorrelator. From Fig. 1(a), we also observe that for any $x_1 > x_2$ there always exists a curve, corresponding to sufficiently high ρ, surrounding the point (x_1, x_2), meaning that the combination lies in the region where MF outperforms MMSE. Notice, on the other hand, that when $x_1 \leq x_2$ the MMSE seems to always outperform the MF (for any ρ); this also includes the case $x_1 = x_2$ of perfect power control. Unfortunately, as we will see, we were not able to prove this last statement analytically.

A. Counterexample

In Fig. 2, we have plotted the relative performance of the three detectors for the case $x_2 = 0.2\, x_1$ and for various values of ρ. Specifically, Fig. 2(a) depicts the case of the MF versus the MMSE detector whereas Fig. 2(b) the MF versus the decorrelator. We note that the MF can significantly outperform both rivals for values of $|\rho| > 0.4$.

It is of interest to consider a specific numerical example. Let $A_1 = 1$, $A_2 = 0.2$ and a noise intensity of -20 dB ($\sigma = 0.1$). A difference of 10–20 dB in user powers and a signal-to-noise ratio (SNR) of the order of 20–30 dB is very common in most simulations in the literature. If $\rho = 0.5$, then the relative performance becomes $P_{\mathrm{MMSE}}/P_{\mathrm{MF}} = 1.6$, $P_D/P_{\mathrm{MF}} = 41.7$. If we now reduce the noise power by 3 dB ($\sigma = 0.071$), then the relative performance becomes $P_{\mathrm{MMSE}}/P_{\mathrm{MF}} = 19.7$, $P_D/P_{\mathrm{MF}} = 838.1$, while with an additional 3-dB reduction in the

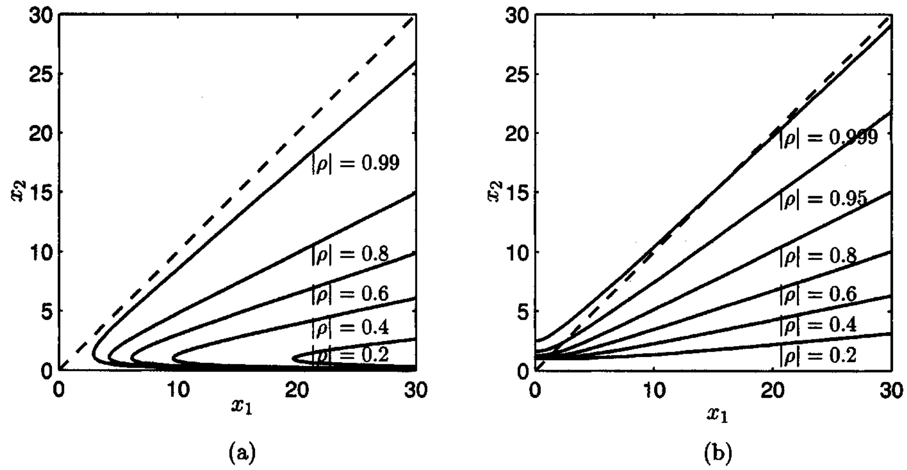

Fig. 1. Combinations of $x_1,\ x_2$ where (a) the MF and the MMSE and (b) the MF and the decorrelator have equal performance, for various values of the cross correlation ρ.

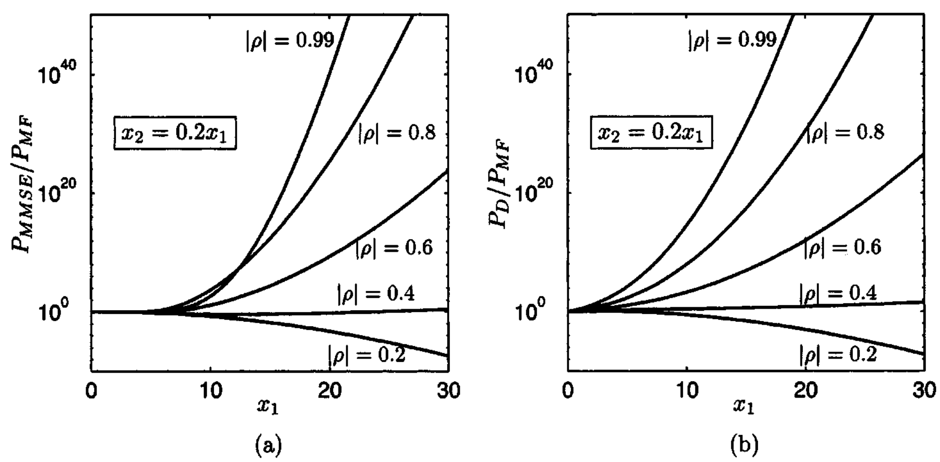

Fig. 2. Relative performance of (a) MF versus MMSE and (b) MF versus the decorrelator, when $x_2 = 0.2\,x_1$ and for various values of ρ.

noise ($\sigma = 0.05$) we obtain an enormous difference $P_{\rm MMSE}/P_{\rm MF} = 5799.3$, $P_D/P_{\rm MF} = 338197.3$.

III. The MMSE Detector Versus the Decorrelating Detector

We now turn, as in [6], to a comparison of the MMSE detector and the decorrelating detector. Here, we restrict attention to the two-user case $(K = 2)$. Proceeding along the same lines as in [6, Proposition 5.2] we will show that the MMSE detector outperforms the decorrelator provided that $|\rho|$ is smaller than some upper limit $\rho_\star < 1$. The significant new information brought by our result as compared to [6] (apart a slight improvement on the upper bound for $|\rho|$) is the fact that the proposed upper bound is tight. By this we mean that if $|\rho| > \rho_\star$ then there are combinations of noise and signal powers for which the decorrelator outperforms the MMSE detector. This, of course, suggests that the conjecture stated in [6] (that the MMSE detector is always better than the decorrelator) is false.

It is convenient to rewrite the error probabilities for the two detectors using a slightly different notation

$$P_{\rm MMSE} = \frac{1}{2}\,Q(ax_1 + b) + \frac{1}{2}\,Q(ax_1 - b) \tag{5}$$

$$P_D = Q(cx_1) \tag{6}$$

where

$$a = \frac{1 - \rho^2 + \phi_2}{\sqrt{(1 + 2\phi_2)(1 - \rho^2) + \phi_2^2}} \tag{7}$$

$$b = \frac{\rho\sqrt{\phi_2}}{\sqrt{(1 + 2\phi_2)(1 - \rho^2) + \phi_2^2}} \tag{8}$$

and $c = \sqrt{1 - \rho^2}$, $\phi_2 = x_2^{-2}$. Notice that $P_{\rm MMSE}$ and P_D are symmetric in the correlation factor ρ. Consequently, without loss of generality, we will assume that $\rho \geq 0$. Let ϕ_2 and ρ be fixed, then parameters $a,\ b,\ c$ become fixed as well since they depend only on ϕ_2 and ρ and not on x_1. The two error probabilities can thus be written as functions of x_1, and we denote their difference by

$$J(x_1) = P_{\rm MMSE}(x_1) - P_D(x_1). \tag{9}$$

In the following subsection we analyze this function in detail.

A. Analysis of $J(x_1)$

We first note that $J(0) = J(\infty) = 0$. Consider now the derivative $J'(x_1)$, or, more precisely, the following expression that has the same sign:

$$\frac{\sqrt{2\pi}}{a}\,e^{c^2x_1^2/2}J'(x_1) = d - F(x_1) \tag{10}$$

where

$$F(x_1) = 0.5\left\{e^{-(c_1x_1+c_2)^2} + e^{-(c_1x_1-c_2)^2}\right\}$$

$d = \frac{c}{a}\,e^{-b^2c^2/2c_1^2}$, $c_1 = \sqrt{a^2 - c^2}$, and $c_2 = ab/c_1$. The number of sign alternations of the expression in (10) for $x_1 \geq 0$ depends on the

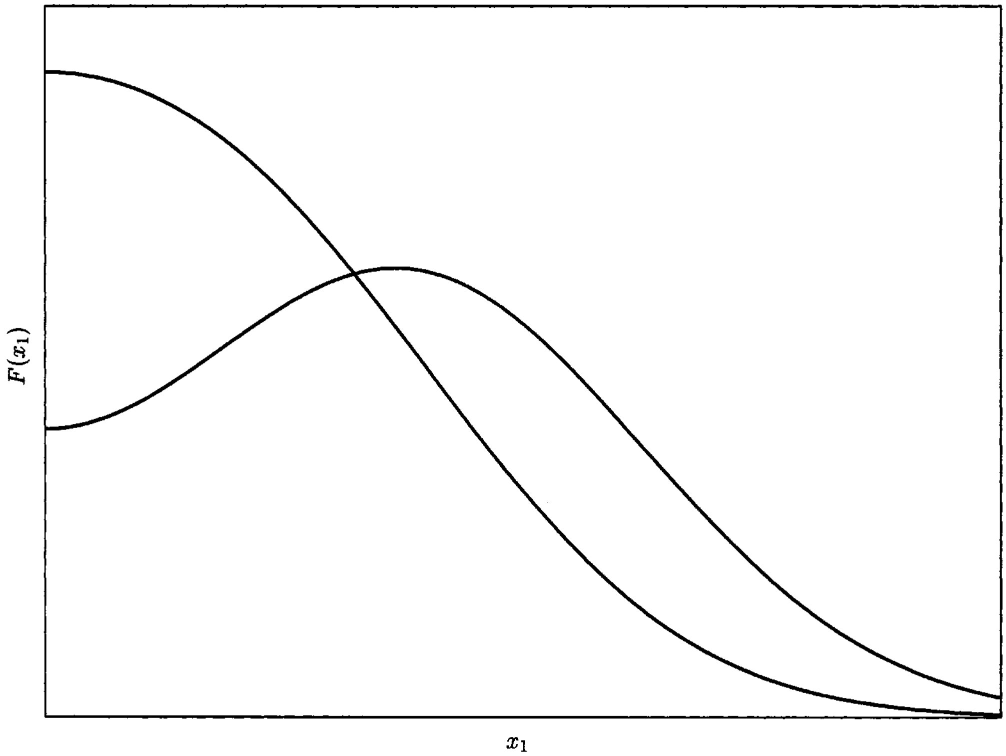

Fig. 3. The two possible forms of the function $F(x_1)$.

number of positive real roots of the equation $d = F(x_1)$. This equation has at least one positive root (because $J(0) = J(\infty) = 0$ guarantees an extremum for $J(x_1)$) and, as one can observe from Fig. 3 where we present the two possible forms of $F(x_1)$, at most two. We can now prove the following lemma.

Lemma: Suppose ϕ_2 and ρ are fixed. Then the MMSE detector is better than the decorrelator for every value of x_1 iff $d \le F(0)$.

Proof: From Fig. 3 we can conclude that if $d \le F(0)$, the equation $d = F(x_1)$ has only one positive root (say, $x_\star$). This means that $J(x_1)$ is decreasing in x_1 for $0 \le x_1 \le x_\star$ and increasing for $x_1 \ge x_\star$. Since $J(0) = J(\infty) = 0$ this suggests that $J(x_1) \le 0$; in other words, that the MMSE detector is superior to the decorrelator for all values of x_1. Consider now the case $d > F(0)$. In order for this to be possible, since we always have a root in our equation, it is clear that $F(x_1)$ must have a minimum at $x_1 = 0$ and a maximum at some positive point (see Fig. 3). We can thus conclude that in this case the MMSE detector can be inferior to the decorrelator for x_1 sufficiently small. □

Let us now substitute all quantities entering in the inequality $d \le F(0)$ in terms of the two parameters ρ and ϕ_2. We then obtain the following equivalent relation:

$$G(\phi_2,\, \rho) = \frac{1-\rho^2+\phi_2}{\sqrt{1-\rho^2}\sqrt{(1+2\phi_2)(1-\rho^2)+\phi_2^2}} \times \exp\left(\frac{-\rho^2\phi_2}{2[(1+2\phi_2)(1-\rho^2)+\phi_2^2]}\right) \ge 1 \quad (11)$$

which due to the Lemma is necessary and sufficient for guaranteeing superior performance for the MMSE detector over the decorrelator for every value of x_1.

From the proof of the Lemma it is clear that if (11) is not satisfied for some combination of ϕ_2 and ρ, we can then find sufficiently small x_1 such that $P_{\mathrm{MMSE}} > P_D$. Let us now examine (11) more closely.

B. Analysis of $G(\phi_2,\, \rho)$

Fix ρ and consider $G(\phi_2,\, \rho)$ as a function of ϕ_2. We can then verify that the sign of its partial derivative with respect to ϕ_2 is the same as the sign of the following third-order polynomial:

$$U(\phi_2,\, \rho) = (1-\rho^2+\phi_2)^3 - \rho^2(1-\rho^2)(1-\rho^2+\phi_2) + 2\rho^2(1-\rho^2)^2. \quad (12)$$

Using standard results concerning roots of third-order polynomials [1, p. 7] one can show that when $\rho > \sqrt{27/28} \approx 0.982$ the polynomial has three distinct real roots, one of which is negative, whereas for $0 < \rho \le \sqrt{27/28}$, the polynomial has one negative real root and two complex conjugate roots.

Consider the case $\rho > \sqrt{27/28}$; as noted above, the polynomial $U(\phi_2,\, \rho)$ has three distinct real roots, one of which is negative and the other two have common sign. Since $U(2(1-\rho^2),\, \rho) < 0$ and

$$\lim_{\phi_2 \to +\infty} U(\phi_2,\, \rho) = +\infty.$$

This means that there is at least one root in the interval $(2(1-\rho^2),\, \infty)$ which, in turn, suggests that the two roots with the common sign are positive. From the above we have the following remarks concerning $G(\phi_2,\, \rho)$.

Remark 1: If $\rho \le \sqrt{27/28}$, then $U(\phi_2,\, \rho)$ is positive for $\phi_2 \ge 0$, and, consequently, $G(\phi_2,\, \rho)$ is strictly increasing in ϕ_2 and larger than unity (since $G(0,\, \rho) = 1$). For this case, the inequality in (11) is clearly satisfied for every $\phi_2 \ge 0$.

Remark 2: If $\rho > \sqrt{27/28}$ then $G(\phi_2,\, \rho)$, as a function of $\phi_2 \ge 0$, presents two local extrema at the two positive real roots of $U(\phi_2,\, \rho)$, the first one being a local maximum and the second a local minimum. Since the first extremum is a local maximum this means that its value is larger than unity (because $G(0,\, \rho) = 1$). The local minimum, however, can be either larger or smaller than unity depending on ρ. If for some ρ

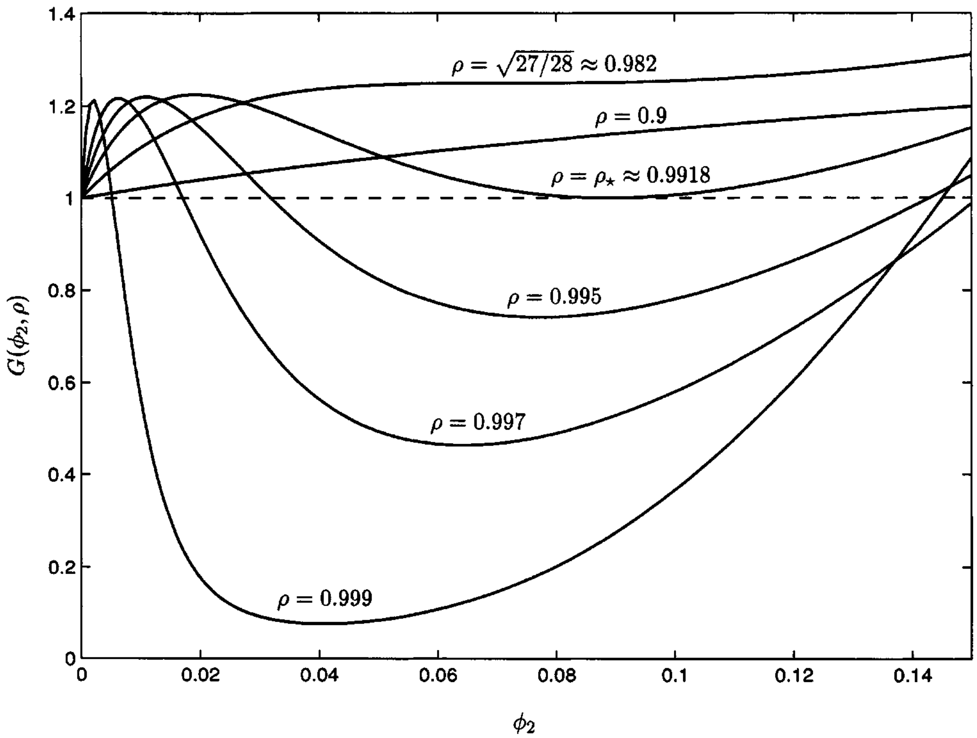

Fig. 4. $G(\phi_2,\, \rho)$ as a function of ϕ_2 for various values of ρ.

the corresponding local minimum of $G(\phi_2,\, \rho)$ goes below unity, then inequality (11) is clearly not satisfied for every $\phi_2 \geq 0$. If, on the other hand, the local minimum is greater or equal than unity then (11) is satisfied for all $\phi_2 \geq 0$.

The variable behavior of $G(\phi_2,\, \rho)$ is depicted in Fig. 4, where we plot this function with respect to ϕ_2 and for characteristics values of the parameter ρ. We are now in a position to find all values of the cross correlation ρ for which the MMSE detector outperforms the decorrelator for all possible noise and signal powers.

Proposition 2: There exists a $\rho_\star$ with $\sqrt{27/28} < \rho_\star < 1$ such that if $0 \leq \rho \leq \rho_\star$ then the MMSE detector outperforms the decorrelator for any value of x_1 and ϕ_2; furthermore if $1 \geq \rho > \rho_\star$ then there exist values for x_1 and ϕ_2 such that the MMSE detector is inferior to the decorrelator.

Proof: Due to Remark 1 we conclude that for any $0 \leq \rho \leq \sqrt{27/28}$ we have that $G(\phi_2,\, \rho) \geq 1$ for all $\phi_2 \geq 0$. This, of course, implies that for any $0 \leq \rho \leq \sqrt{27/28}$ the MMSE detector outperforms the decorrelator for any noise and signal powers. With the help of Remark 2 we will be able to slightly improve this result and propose an upper bound on ρ that is in fact tight.

Consider the case $1 \geq \rho > \sqrt{27/28}$ where Remark 2 applies. If we plot the local minimum of $G(\phi_2,\, \rho)$ as a function of ρ then we can observe that it is decreasing with ρ. Furthermore, there exists a value $\rho_\star$ for which the corresponding local minimum is exactly equal to unity. Due to Remark 2 and the monotonicity of the local minimum as a function of ρ we then conclude that for $\rho_\star \geq \rho > \sqrt{27/28}$ we have $G(\phi_2,\, \rho) \geq 1$ for all values of ϕ_2. On the other hand, if $1 \geq \rho > \rho_\star$, there exist values for ϕ_2 where the inequality $G(\phi_2,\, \rho) \geq 1$ is false (see Fig. 4).

Combining the two intervals we conclude that for any $0 \leq \rho \leq \rho_\star$ the inequality in (11) is true for all values of ϕ_2, therefore the MMSE detector outperforms the decorrelator for any noise and signal powers. The upper limit $\rho_\star$ can be computed numerically; the value we obtain is $\rho_\star = 0.991765239964$. From the above discussion we can also deduce that the proposed upper limit $\rho_\star$ is tight since for any $1 \geq \rho > \rho_\star$ we can find values for the parameter ϕ_2 such that (11) is false, meaning that for sufficiently small x_1 we have $J(x_1) > 0$. □

C. Counterexample

A counterexample for the conjecture in [6] is the following. Let $x_1 = 1$, $x_2 = 5$, and $\rho = 0.996 > \rho_\star$ then $P_{\rm MMSE} = 0.4699 > P_D = 0.4644$. It should be mentioned, however, that unlike the case presented in the previous section, the counterexample here is of no practical importance. For cross correlations as large as 0.9999, the error rates, where the decorrelator is better than MMSE, were always above 0.4.

IV. Power Control

In this section we to consider the special case $A_1 = A_2 = \cdots = A_K = A$ or, equivalently, $x_1 = x_2 = \cdots = x_K = x$ which corresponds to perfect power control. The error probabilities of the three detectors, for this case, take the form

$$P_{\rm MMSE} = 2^{1-K} \sum_{b_i=\pm 1} Q\left(x \frac{e_1^t (x^2 R + I)^{-1} Rb}{\sqrt{z(R,\, x)}}\right) \qquad (13)$$

$$P_D = Q\left(\frac{x}{\sqrt{e_1^t R^{-1} e_1}}\right) \qquad (14)$$

$$P_{\rm MF} = 2^{1-K} \sum_{b_i=\pm 1} Q(x e_1^t Rb) \qquad (15)$$

where $z(R,\, x) = e_1^t (x^2 R + I)^{-1} R (x^2 R + I)^{-1} e_1$.

A. MMSE Versus the Decorrelator

Here we will examine the relative performance of the two detectors for the two-user case. We have the following proposition.

Proposition 3: For the two-user case and under perfect power control the MMSE detector always outperforms the decorrelator.

Proof: Denote the two error probabilities as $P_{\rm MMSE}(x,\,\rho)$ and $P_D(x,\,\rho)$. Since $P_{\rm MMSE}(x,\,1) = P_D(x,\,1)$, consider $\rho < 1$. Applying the change of variables $x = \xi/\sqrt{1-\rho^2}$ the error probabilities become

$$P_{\rm MMSE}(\xi,\,\rho) = 0.5\,Q\left(\xi\frac{\xi^2+1+\rho}{\sqrt{z(\rho,\,\xi)}}\right) + 0.5\,Q\left(\xi\frac{\xi^2+1-\rho}{\sqrt{z(\rho,\,\xi)}}\right) \tag{16}$$

$$P_D(\xi,\,\rho) = Q(\xi), \tag{17}$$

where

$$z(\rho,\,\xi) = \xi^4 + (1-\rho^2)(2\xi^2+1).$$

We will now show that $P_{\rm MMSE}(\xi,\,\rho)$, for fixed ξ, is a decreasing function in ρ. This is sufficient to prove the proposition since then

$$P_{\rm MMSE}(\xi,\,\rho) \le P_{\rm MMSE}(\xi,\,0) = Q(\xi) = P_D(\xi,\,\rho).$$

To show that the partial derivative of $P_{\rm MMSE}(\xi,\,\rho)$ with respect to ρ is negative, after some tedious but straightforward calculations, is equivalent to showing the following inequality:

$$\rho\frac{2\xi^2+1}{\xi^2+1} \ge \tanh\left(\rho\frac{\xi^2(1+\xi^2)}{\xi^4+(1-\rho^2)(2\xi^2+1)}\right). \tag{18}$$

For $\rho \ge (\xi^2+1)/(2\xi^2+1)$ this inequality is true because for positive z we have $1 \ge \tanh(z)$. On the other hand, when $\rho \le (\xi^2+1)/(2\xi^2+1)$, since $\rho^2 \le \rho$ and $z \ge \tanh(z)$ for $z \ge 0$; we can write

$$\begin{aligned}\rho\frac{2\xi^2+1}{\xi^2+1} &\ge \rho \ge \tanh(\rho)\\ &\ge \tanh\left(\rho\frac{\xi^2(1+\xi^2)}{\xi^4+(1-\rho^2)(2\xi^2+1)}\right)\end{aligned} \tag{19}$$

which completes the proof. □

B. MF Versus the MMSE Detector

In view of the previous result and also the fact that, from Fig. 1(a), line $x_1 = x_2$ lies entirely in the area where the MMSE detector outperforms the MF filter, one might conjecture that the MMSE detector would be superior to the MF for the power control case. Unfortunately, it turns out that even this conjecture is false as one can find counterexamples for odd K.

Proposition 4: Let the number of users K be odd and the corresponding signature waveforms be equicorrelated with common correlation ρ. Then under perfect power control and for any $x \ge \sqrt{2}$ there exists correlation ρ sufficiently close to unity such that the MF outperforms the MMSE detector.

Proof: The correlation matrix, for equicorrelated waveforms, takes the form $R = (1-\rho)I + \rho VV^t$ where $V = [1\ \ldots\ 1]^t$. The corresponding error probabilities then become functions of x and ρ and let us denote them as $P_{\rm MMSE}(x,\,\rho)$ and $P_{\rm MF}(x,\,\rho)$.

The two error probabilities satisfy $P_{\rm MMSE}(x,\,1) = P_{\rm MF}(x,\,1)$. Since both functions are continuous with respect to their arguments, it is sufficient to show that for fixed $x \ge \sqrt{2}$ we have

$$\left.\frac{\partial P_{\rm MMSE}(x,\,\rho)}{\partial\rho}\right|_{\rho=1} \le \left.\frac{\partial P_{\rm MF}(x,\,\rho)}{\partial\rho}\right|_{\rho=1}. \tag{20}$$

Due to the special structure of the correlation matrix, the sums in the definition of the two error probabilities reduce to sums containing only K terms instead of the 2^{K-1} [6]. If we compute the two partial derivatives for $\rho = 1$, then inequality (20) is equivalent to

$$\sum_{n=0}^{r} e^{-x^2(K-2n)^2/2}\binom{K}{n}\left\{1 - \frac{(K-2n)^2}{K}(1+rx^2)\right\} < 0 \tag{21}$$

where $K = 2r+1$. Condition (21) can be seen to be true for $x^2 > 2$, since it makes the quantity in the brackets negative for all $n = 0,\,\ldots,\,r$. This concludes the proof. □

Although Proposition 4 indicates that it is possible for the case of odd K to find counterexamples to the conjecture that the MMSE detector is better than the MF, we should nevertheless note that, at least for the equicorrelated case, these counterexamples appear only at extreme values of the cross correlation ρ. Specifically, it was found numerically that for $K = 3$ the cross correlation must exceed the value 0.9942 while things seem to improve (for MMSE detector) as K increases since for $K = 5$ the cross correlation needs to be larger than 0.9975 and for $K = 7$ larger than 0.9986.

C. MF Versus the Decorrelator

As we stated in Section II, it is known [8, p. 249] that the MF outperforms the decorrelator for sufficiently high noise power (provided that User 1 is not orthogonal to the interfering users). Since this fact holds for any combination of $x_1,\,x_2$ it is also true for the perfect power control case. If we are, however, interested in the case where the noise power does not go to infinity (i.e., x does not tend to zero), then it is possible to obtain a result similar to the one presented in the previous subsection (in fact, slightly stronger). Again if we denote the corresponding error probabilities with $P_{MF}(x,\,\rho)$ and $P_D(x,\,\rho)$, then we obtain $P_D(x,\,1) = 0.5$ and

$$P_{\rm MF}(x,\,1) = 2^{1-K}\sum_{n=0}^{K-1}\binom{K-1}{n}Q(x[K-2n]). \tag{22}$$

Using the property that for positive $x,\,y$ we have

$$Q(x+y) + Q(x-y) < Q(y) + Q(-y) = 1$$

it is easy to show that $P_{\rm MF}(x,\,1) < 0.5$. Because of continuity this also suggests that for any fixed $x > 0$ we can find ρ sufficiently close to unity such that $P_{\rm MF}(x,\,\rho) < P_D(x,\,\rho)$.

V. Conjectures

Next we present a number of statements, in the form of conjectures, concerning comparisons of the three detectors under signaling conditions that constitute generalization to the ones presented in the previous sections. It should be noted that these conjectures are supported by extensive numerical computations of the corresponding error rates. Unfortunately, up to now it was not possible to prove them analytically.

The first conjecture refers to the comparison of the MMSE and the decorrelator and constitutes the generalization of Proposition 2 to the K-user and equicorrelated signals case.

Conjecture 1: Proposition 2 extends to arbitrary number of users K with equicorrelated signals. The upper limit $\rho_\star(K)$ increases with K.

Conjecture 2: Proposition 3 extends to arbitrary number of users and arbitrary crosscorrelation matrix R.

The next conjecture refers to the comparison of the MMSE detectorand the MF detectors under perfect power control. With Proposition 4, we have seen that, when the number of users K is odd then there exist examples where the MF outperforms the MMSE detector. Such example was impossible to find for even values of K. In fact, there are strong indications that for this case the MMSE detector is uniformly better than the MF (notice in Fig. 1(a) that the dashed line $x_1 = x_2$ lies in the region where the MMSE detector is better than the MF).

Conjecture 3: Under perfect power control and equicorrelated signals the MMSE detector outperforms the MF when the number of users K is even.

VI. CONCLUSION

The results described in this correspondence have been pursued primarily out of theoretical interest. The significant practical advantages of the MMSE detector over the decorrelator and the MF would likely outweigh any performance disadvantage revealed here, inasmuch as the range of parameters for which the performance disadvantages arise are somewhat at the extremes for practical systems. Nevertheless, these results do provide some cautionary guidance concerning the relative merits of linear multiuser detectors.

APPENDIX

Proof of Proposition 1: The error probabilities for the detectors of interest are given in (2)–(4). It is convenient, however, to rewrite these expressions in order to reveal the linear dependency on x_1 of the arguments of the Q-functions. Since $X^{-2} = x_1^{-2}e_1e_1^t + S$ with $S = \operatorname{diag}\{0, x_2^{-2}, \ldots, x_K^{-2}\}$ the matrix inversion lemma yields

$$(R + X^{-2})^{-1}e_1 = \frac{(R+S)^{-1}e_1}{1 + x_1^{-2}e_1^t(R+S)^{-1}e_1}. \tag{23}$$

Substituting this into (2) and noting that

$$e_1^t(R+S)^{-1}Re_1 = e_1^t(R+S)^{-1}(R+S-S)e_1 = 1 \tag{24}$$

(because $Se_1 = 0$), we obtain the following alternative expression:

$$P_{\mathrm{MMSE}} = 2^{1-K}\sum_{b_i=\pm 1} Q\left(\frac{x_1 + e_1^t(R+S)^{-1}R\tilde{X}b}{\sqrt{e_1^t(R+S)^{-1}R(R+S)^{-1}e_1}}\right) \tag{25}$$

where $\tilde{X} = \operatorname{diag}\{0, x_2, \ldots, x_K\}$. It is also convenient to explicitly display the dependence of the MF detector's performance on x_1; namely,

$$P_{\mathrm{MF}} = 2^{1-K}\sum_{b_i=\pm 1} Q(x_1 + e_1^tR\tilde{X}b). \tag{26}$$

It should be noted that, except of course of x_1, none of the remaining quantities in (25) and (26) depends on x_1.

Using (24), the fact that $e_1^tRe_1 = 1$ and the Schwarz inequality we can show that

$$\frac{1}{\sqrt{e_1^t(R+S)^{-1}R(R+S)^{-1}e_1}} \le 1 \tag{27}$$

with equality iff

$$R^{1/2}e_1 = \alpha R^{1/2}(R+S)^{-1}e_1 \tag{28}$$

for some scalar α. It is easy to verify that (28) holds iff $\rho_{l1} = 0$, $l = 2, \ldots, K$, where ρ_{l1} is the lth component of the first column of R. In other words, we have equality in (27) iff simultaneously all interfering users have signature waveforms that are orthogonal to the signature waveform of User 1. If at least one interfering user does not satisfy this constraint, then the inequality in (27) is strict.

Now, from (25) and (26), it follows immediately that, with (27) strict and for all sufficiently large x_1, we will have $P_{\mathrm{MMSE}} > P_{\mathrm{MF}}$. This is because each of these error probabilities is dominated in the tails (of large values of x_1) by the term involving the Q-function with the smallest argument. If (27) is strict, then for sufficiently large x_1 the smallest such argument of P_{MMSE} will be smaller than the smallest such argument in P_{MF}.

As far as the relative performance of the MF and the decorrelator is concerned, it is known [8, p. 255] that for sufficiently high noise power the MF outperforms the decorrelator. What our proposition suggests is that the MF also outperforms the decorrelator when the signal power of the user of interest is sufficiently high. Indeed, notice that by using again the Schwarz inequality we can show

$$1 = e_1^tR^{-1/2}R^{1/2}e_1 \le \sqrt{e_1^tR^{-1}e_1}\sqrt{e_1^tRe_1} = \sqrt{e_1^tR^{-1}e_1} \tag{29}$$

with equality iff $\alpha e_1 = Re_1$ for some scalar α or equivalently iff $\rho_{1l} = 0$, $l = 2, \ldots, K$. The rest of the proof goes exactly as in the previous case. □

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