where $E_f = \int_{-\infty}^{\infty} |f(t)|^2 \, dt$ is the signal energy. Note that the coefficients a'_{kn} in (9) are obtainable in explicit form as the coefficients in the binomial expansion of

$$\left(1 - \alpha'_1 Z - \alpha'_2 Z^2\right)^n = 1 - \sum_{k=1}^{2n} a'_{kn} Z^k$$

where $\alpha'_1 = 2 - \sqrt{2}$ and $\alpha'_2 = 1 - \sqrt{2}$.

III. RANDOM CASE

In this section $x(t)$ is assumed to be a complex-valued WSS process with power spectral density $S_x(\omega) = 0$ for $|\omega| > \pi$, and $R_x(\tau) = (1/2\pi)\int_{-\pi}^{\pi} S_x(\omega) e^{j\omega\tau} \, d\omega$ for all τ, where $R_x(\tau) \triangleq E\{x(t+\tau)x^*(t)\}$. With $I_n(\omega) \triangleq 1 - \sum_{k=1}^{n} a_{kn} e^{-j\omega T}$ for $\omega \in [-\pi, \pi]$ it is an easy exercise to verify that

$$E\left\{\left| x(t) - \sum_{k=1}^{2n} a_{kn} x(t - kT) \right|^2 \right\}$$
$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} S_x(\omega) |I_{2n}(\omega)|^2 \, d\omega.$$

From the preceding section we have

$$\left(1 - \alpha_1 e^{-j\omega T} - \alpha_2 e^{-j2\omega T}\right)^n = 1 - \sum_{k=1}^{2n} a_{kn} e^{-jk\omega T} = I_{2n}(\omega)$$

thus

$$|I_{2n}(\omega)|^2 = \left(\left|1 - \alpha_1 e^{-j\omega T} - \alpha_2 e^{-j2\omega T}\right|^2\right)^n.$$

However, for $\alpha_1 = \alpha'_1 = 2 - \sqrt{2}$ and $\alpha_1 = \alpha'_2 = 1 - \sqrt{2}$, $0 < T \leq 1/2$, and $\omega \in [-\pi, \pi]$, we have $|I_{2n}(\omega)|^2 \leq \beta^n$ where $\beta = 4(3 - 2\sqrt{2}) \doteq 0.6863$, since $|1 - \alpha'_1 e^{-j\omega T} - \alpha'_2 e^{-j2\omega T}|^2 \leq \beta$ for $-\pi/2 \leq \omega T \leq \pi/2$.

Theorem 2: For $x(t)$ a WSS random process with power spectral density having support on $[-\pi, \pi]$ and $0 < T \leq 1/2$ the coefficients $\{a'_{kn}\}$ determined from the identity

$$\left[1 - (2 - \sqrt{2}) e^{-j\omega T} - (1 - \sqrt{2}) e^{-j2\omega T}\right]^n = 1 - \sum_{k=1}^{2n} a'_{kn} e^{-jk\omega T}$$

are such that

$$E\left\{\left| x(t) - \sum_{k=1}^{2n} a'_{kn} x(t - kT) \right|^2 \right\} \leq \beta^n \cdot \frac{1}{2\pi} \int_{-\pi}^{\pi} S_x(\omega) \, d\omega$$

$$= \beta^n \cdot R_x(0)$$

where $\beta = 0.6863$.

Again we observe that the mean-square convergence is uniform in t for $t \in (-\infty, \infty)$ and the coefficients $\{a'_{kn}\}$ are independent of both T and the spectral properties of the particular random process being estimated.

IV. REMARKS

We have determined a set of numerical coefficients which yields a one-step prediction of either a deterministic or random band-limited process with error that decreases geometrically as the number of past samples used becomes infinite. While the deterministic signal was assumed to have finite energy, the assumption that the signal spectrum is absolutely integrable works equally well for

$$\left| f(t) - \sum_{1}^{2n} a_{kn} f(t - kT) \right| = \left(\frac{1}{2\pi}\right) \cdot \left| \int_{-\pi}^{\pi} F(\omega) I_{2n}(\omega) \, d\omega \right|$$

$$\leq \frac{1}{2\pi} \int_{-\pi}^{\pi} |F(\omega)| \cdot |I_{2n}(\omega)| \, d\omega$$

$$\leq \beta^{n/2} \cdot \frac{1}{2\pi} \int_{-\pi}^{\pi} |F(\omega)| \, d\omega,$$

where $|I_{2n}(\omega)|^2 \leq \beta^n$ as in the previous section.

Clearly, the technique used here can be extended using three parameters to introduce another degree of freedom:

$$|1 - \alpha_1 e^{-j\omega T} - \alpha_2 e^{-j2\omega T} - \alpha_3 e^{-j3\omega T}|^n = \left|1 - \sum_{k=1}^{3n} a_{kn} e^{-jk\omega T}\right|.$$

Using this approach with normalized band-width we have found explicit coefficients $\{a_{kn}\}$ such that the prediction error goes to zero at a geometric rate for intersample spacings T satisfying $0 < T \leq 2/3$; that is, the past samples need only be taken at a rate equal to or in excess of $3/2$ the Nyquist rate. Details of this approach and numerical examples are currently in preparation.

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Detection and Diagnosis of Abrupt Changes in Modal Characteristics of Nonstationary Digital Signals

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Abstract—New "instrumental" tests for detecting and diagnosing changes in the poles of a signal having unknown time-varying zeros are proposed. Numerical results for nonstationary scalar signals are given. The extension of these tests to the vector case may be used for vibration monitoring.

I. INTRODUCTION

The problem of detecting changes in spectral properties arises frequently in practice, in the segmentation of nonstationary digital signals or the monitoring of time series, for example. Most of the solutions, which are available so far in the time domain basically use the complete set of the (known or identified) model parameters: this is the case for innovations-based detectors and standard likelihood ratio tests. For example, [1] presented a new on-line algorithm based upon Kullback's divergence between the conditional probability laws of the observations under a "reference" model and a "current" one.

In some practical situations, one is interested in detecting changes in a subset of the model parameters, while the complementary subset of model parameters are completely unknown and thus have to be considered as nuisance parameters. This is the case, for example, in vibration monitoring, where one wishes to detect changes in vibrating characteristics of systems subject to nonstationary unknown excitation such as swell, wind, or earthquakes. In this case, the change detection problem can be formulated as follows: using an autoregressive moving-average (ARMA) model with (highly) nonstationary unknown moving-average (MA) coefficients to model the excitation [12], detect a change in the autoregressive (AR) part (assumed stationary) and, if possible, determine which AR coefficients or which poles have changed (this latter task is the diagnosis problem). Let us em-

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phasize that, even in the scalar case, this diagnosis problem is not standard.

Because of the highly time-varying character of the unknown MA coefficients, none of the standard elimination methods for nuisance parameters [3] seems to be of help for this change detection problem. On the other hand, it has been shown in [4] that it is possible to obtain consistent estimates of the AR

$$\mathscr{H}_{p,N-1}(s) = \begin{bmatrix} R_{q-p+1}(s) & R_{q-p+2}(s) & R_{q+1}(s) & R_{q+2}(s) & \cdots & R_{N+q-p}(s) \\ R_{q-p+2}(s) & & & & & \\ R_{q-p+3}(s) & & & & & \\ \vdots & & & & & \\ R_{q+1}(s) & R_{q+2}(s) & & & & R_{N+q}(s) \end{bmatrix}$$

coefficients without knowing (or using estimates of) the varying MA coefficients.

The basic idea underlying this correspondence is thus the following: instead of using standard likelihood ratio methods which are of no help in the present case because of the unknown MA part, one may base the detection upon the same idea used for the identification. Using this idea, a statistic U is introduced and the central limit theorem is shown to hold [8] for this statistic U under both hypotheses: null H_0 (i.e., no change) and local alternative H_1 (i.e., small change). This gives a test statistic for a global test (for a change in the AR part) with no diagnosis about the nature of the change. Using the effect of specific parameter changes, such as, for example, changes in poles or vibrating modes, on the mean of U under H_1, one may design specific tests for monitoring vibrating modes separately. We will present another approach for solving the diagnosis problem, which consists basically of reidentifying each pole which has to be monitored.

In this correspondence, we investigate only the scalar case. The extension of the proposed test to the vector case is reported elsewhere [2]; let us only emphasize that this extension may be used, for example, to solve the problem of vibration monitoring for offshore platforms.

Sections II and III present the proposed off-line algorithms for the two problems of interest—detection and diagnosis—while numerical results are given in Section IV. The performance in the simple case of changes in AR models (no MA part) are investigated in Table II, with special emphasis on the problem of coupling effects that arise during diagnosis upon the poles. The performance of the proposed tests in simulations with scalar ARMA signals is reported in Table III for the case where the nonstationary MA part is piecewise constant.

This numerical analysis is the only justification of the two proposed methods for solving the diagnosis problem; this is not the case for the detection problem for which a theoretical basis can be found in [8]. Section V outlines the main conclusions of this study.

II. DETECTION OF CHANGES IN THE AR PART OF AN ARMA MODEL WITH NONSTATIONARY UNKNOWN MA COEFFICIENTS

As mentioned in the Introduction, this correspondence is focused on the scalar case. Thus let us consider the following model:

$$y_t = \sum_{i=1}^{p} a_i y_{t-i} + \sum_{j=0}^{q} b_j(t) e_{t-j}, \tag{1}$$

where $(e_t)_t$ is a Gaussian white noise with constant variance σ_e^2 and $p \geq q$.

Here the unknown MA coefficients (b_j) are time-varying and may even be subject to jumps. The problem to be solved is the (off-line) detection of abrupt changes or jumps in the AR parameters (a_i). We shall first recall the main results concerning the identification problem because, as mentioned in the Introduction, it is the starting point of our detection procedure.

A. Identification of the AR Coefficients Without Knowing the Nonstationary MA Coefficients

Assume that a single record $(y_0, \cdots, y_s)$ of the process (y_t) is available. The so-called instrumental variable method [13] for identification has been recently proved [4] to provide consistent estimates of the AR parameters in the present framework. More precisely, let

the $(p + 1) \times N$ empirical Hankel matrix of the process $(y_t)_{0 \leq t \leq s}$, where $N \geq p$ is the number of instruments and

$$R_k(s) = \sum_{t=0}^{s-k} y_{t+k} y_t; \qquad k \geq 0.$$

Then the least-squares solution $(\hat{a}_p(s), \hat{a}_{p-1}(s), \cdots, \hat{a}_1(s))$ of

$$\begin{pmatrix} -a_p & -a_{p-1} & \cdots & -a_1 & 1 \end{pmatrix} \mathscr{H}_{p,N-1}(s) = 0$$

is a consistent estimate of true vector parameter

$$\theta = \begin{pmatrix} a_p & a_{p-1} & \cdots & a_1 \end{pmatrix}'$$

of model (1). See [4] for a complete proof and precise statement of the consistency result. This result does not require any stationarity assumption about the MA parameters $b_j(t)$. In this sense, this identification method of the AR part may be thought of as being robust with respect to the unknown MA part. Good numerical results in which high-order modes are correctly identified have been obtained with $N \simeq 3p$ for offshore platform data [12].

B. The Change Detection Problem

The use of standard observation-based likelihood ratio techniques for solving this problem would require either an identification of the MA coefficients $b_j(t)$ using, for example, a forgetting factor, or maximization or integration of the likelihood with respect to a prior distribution of these unknown parameters [3]. Because of their highly varying features (related, for example, to the shock or turbulence effects of the sea on an offshore platform), these approaches do not seem to be appropriate. (Recall also that in [5] Bohlin assumed that convenient values of the MA coefficients were available.)

Moreover, the Fisher information matrix of an ARMA model is not block diagonal: an interaction takes place between the AR and the MA coefficients. In other words, a coupling effect exists between the detection of changes in poles or zeros, and therefore the use of (local) likelihood methods as in [7], [9], and [11] for detecting changes on poles is not convenient when the zeros have to be viewed as nuisance parameters.

Keeping in mind the "robustness" properties of the identification procedure with respect to the nuisance parameters, we propose the following off-line change detection procedure. Let us now assume that a reference model parameter $\theta_0 = (a_p^0 \cdots a_1^0)'$ has been estimated on a record of signals y, and let us consider the following problem: given a new record of signals y, decide whether they follow the same model or not. We propose the following solution: compute again the empirical Hankel matrix $\mathscr{H}_{p,N}$ corresponding to this new record, and look at the "size" of the vector U_N defined by

$$U_N'(s) = \begin{pmatrix} -a_p^0 & \cdots & -a_1^0 & 1 \end{pmatrix} \mathscr{H}_{p,N-1}(s). \tag{4}$$

If no change has occurred in the AR part, this U vector should be close to zero; in case of a change in the AR parameters, this U vector should be significantly different from zero.

Let us rewrite $U_N(s)$ in a numerically more efficient way, as

$$U_N(s) = \sum_{t=q+N}^{s} w_t Z_t, \tag{5}$$

where

$$w_t = y_t - a_1 y_{t-1} - \cdots - a_p y_{t-p} \tag{6}$$

is the MA part and

$$Z_t = (y_{t-q-1} \quad y_{t-q-2} \cdots y_{t-q-N})'. \tag{7}$$

Under the hypothesis of no change (i.e., θ_0 still represents the AR part of the actual process), Z_t is orthogonal to w_t, and the covariance matrix of U is

$$\Sigma_N(s) = \sum_{t=q+N}^{s-q} \sum_{i=-q}^{q} \mathbb{E}_{\theta_0}(w_t w_{t-i} Z_t Z'_{t-i}), \tag{8}$$

because, for $|t - r| \geq q + 1$,

$$\mathbb{E}_{\theta_0}(w_t \quad w_r \quad Z_t \quad Z'_r) = 0.$$

Finally, let $\hat{\Sigma}_N(s)$ be the following matrix:

$$\hat{\Sigma}_N(s) = \sum_{t=q+N}^{s-q} \sum_{i=-q}^{q} w_t w_{t-i} Z_t Z'_{t-i}. \tag{9}$$

Despite the fact that the process y_t, and thus Z_t, is nonstationary, the two following theorems hold [8].

Nonstationary Law of Large Numbers: $\hat{\Sigma}_N$ is a consistent estimate of Σ_N, namely,

$$\Sigma_N^{-1}(s)\hat{\Sigma}_N(s) \xrightarrow[s \to \infty]{i.p.} I_N \tag{10}$$

under both the null hypothesis (the set of AR parameters is θ_0) and the local alternative hypothesis (the set of AR parameters is $\theta_0 + (\delta\theta/\sqrt{s})$, where $\delta\theta$ is fixed).

Central Limit Theorem: Under the probability law $\mathbb{P}_{\theta_0}$, we have

$$\Sigma_N(s)^{-1/2} \cdot U_N(s) \xrightarrow[s \to \infty]{\mathcal{D}} \mathcal{N}(0, I_N), \tag{11}$$

and under the "small" change hypothesis $\mathbb{P}_{\theta_0 + (\delta\theta/\sqrt{s})}$ we have

$$\Sigma_N(s)^{-1/2} \cdot \left(U_N(s) - \mathcal{H}'_{p-1, N-1} \frac{\delta\theta}{\sqrt{s}}\right) \xrightarrow[s \to \infty]{\mathcal{D}} \mathcal{N}(0, I_N). \tag{12}$$

The proofs of the theorems are based upon extensive use of various limit theorems for martingales. Because of these results, the use of the local approach for detecting changes [7], [9], [11] reduces the original problem to that of detecting a change in the mean value of a Gaussian process.

We will make extensive use of the following general result. Assume U is (asymptotically) distributed as $\mathcal{N}(0, \Sigma)$ under H_0 and as $\mathcal{N}(\mu, \Sigma)$ under H_1. For testing $\mu = 0$ against $\mu \neq 0$, asymptotically one computes

$$U'\Sigma^{-1}U \tag{13}$$

and compares it to a threshold. On the other hand, for testing $\mu = 0$ against $\mu \in$ range (A), where A is a full column rank matrix, one computes

$$U'\Sigma^{-1}A(A'\Sigma^{-1}A)^{-1}A'\Sigma^{-1}U, \tag{14}$$

which is nothing but the maximum value, with respect to ν, of the log likelihood ratio between H_0 and H_1 with $\mu = A\nu$.

Thus for small changes in θ, using (12) and (14), we get the following χ^2 test:

$$t_0 = U'_N \Sigma_N^{-1} \mathcal{H}'_{p-1, N-1} \left(\mathcal{H}_{p-1, N-1} \Sigma_N^{-1} \mathcal{H}'_{p-1, N-1}\right)^{-1}$$
$$\cdot \mathcal{H}_{p-1, N-1} \Sigma_N^{-1} U_N. \tag{15}$$

As the true covariance matrix Σ_N of U_N is not known, in practice we use an estimate for computing t_0. The estimate $\hat{\Sigma}_N$ given by (9), which is consistent by (10), is a possible choice. Another choice is shown in (19).

C. Further Results for Some Special Cases

1) Let us first investigate the AR case where $q = 0$. Then

$$U_N(s) = \sum_{t=N}^{s} w_t Z_t$$

where

$$w_t = b_0(t)e_t$$

and

$$Z_t = (y_{t-1} \cdots y_{t-N})'. \tag{16}$$

Furthermore,

$$\Sigma_N(s) = \sigma_e^2 \sum_{t=N}^{s} b_0^2(t)\mathbb{E}_{\theta_0}(Z_t Z'_t),$$

where $\mathbb{E}_{\theta_0}(Z_t \quad Z'_t)$ is the Toeplitz covariance matrix of size N of the observation process $(y_t)_t$.

Let us now consider the stationary AR case, that is, let us assume that $b_0(t)$ is constant, which is usually the case when one is interested in changes in AR parameters. Then Σ_N may be estimated by

$$\hat{\Sigma}_N(s) = b_0^2 \sigma_e^2 \sum_{t=N}^{s} Z_t Z'_t. \tag{17}$$

Furthermore, let us assume that $N = p$, which is the minimum number of instruments to be used. Then the empirical Hankel matrix $\mathcal{H}_{p-1, p-1}$ is invertible, and the global test t_0 (15) is

$$t_0 = U'_p \hat{\Sigma}_p^{-1} U_p, \tag{18}$$

which is nothing but the classical local likelihood ratio test (third version of the cusum type algorithm derived by Nikiforov) for detecting changes in AR coefficients [10], [11].

2) Let us now consider the ARMA $(p, p - 1)$ case, which naturally arises from state space models without observation noise; this model has been used for the vibration monitoring application on offshore platforms [12]. Then

$$U_N(s) = \sum_{t=N+p-1}^{s} w_t Z_t,$$

with w_t as in (6) and

$$Z_t = (y_{t-p} \quad y_{t-p-1} \cdots y_{t-N-p+1})'.$$

For $N = p$, the global test t_0 (15) is still as in (18).

Notice that, instead of the estimate (9) of Σ_N, one can compute another "approximate" estimate:

$$\tilde{\Sigma}_N(s) = \left(\sum_{t=p}^{s} w_t^2\right)\left(\sum_{t=N+p-1}^{s} Z_t Z'_t\right) \Big/ (s - p), \tag{19}$$

[see (17)] which leads to a global test numerically better conditioned than the initial one, although we have no theoretical justification for it. However, it can be proved theoretically that $\tilde{\Sigma}_N(s)$ is always invertible (even if p is not the correct AR order) and that, under general conditions [8], $\Sigma_N(s)$ is invertible provided that the AR order is not underestimated. Finally, let us mention that all these tests may be extended to the vector case. (See [2].)

III. DETECTION WITH DIAGNOSIS

Here we investigate the problem of detecting changes in the AR part, with diagnosis upon which AR coefficients or which poles have changed, and still without knowing the nonstationary

MA coefficients. As in Section II, we only investigate the scalar case. Let us first emphasize that, even in the stationary AR case, this diagnosis problem is not so much standard, especially when the poles are of interest. As far as we know, the only approach which has been investigated for solving this problem is the so-called multiple-model approach described, for example, in [15] and [16]. Two approaches are presented in this section: a sensitivity method which looks for changes (on the AR parameters or on the poles) constrained into a subspace, and a decoupling method which is a kind of filter bank approach and which basically reidentifies each pole to be monitored.

A. Sensitivity Method

It has been shown in the previous section that a possible solution to the problem of detecting changes in the AR parameters without knowing the MA ones is to solve the equivalent Gaussian testing problem for the instrumental statistic U. (Recall (11) and (12), which summarize the nonstationary central limit theorem.)

The basic idea underlying the sensitivity method is to take into account the effect $\delta\theta$ of changes of interest (for example on separate poles) on the θ parameter (3) and to use the same likelihood ratio approach based upon the U vector.

Describing the diagnosis problem more precisely, let ψ be the m-dimensional set of the "free" parameters to be monitored, and let ψ_0 be the set of their nominal values. Then changes $\delta\psi$ in these free parameters induce changes in the AR parameters $\delta\theta$ given by

$$\delta\theta = f(\delta\psi)$$

where f is a nonlinear differentiable function. Let $J = f'(\psi_0)$ be the $p \times m$ Jacobian matrix,

$$J = \left(\frac{\partial a_i}{\partial \psi_j}\right)_{\{\psi_j\} = \psi_0} . \qquad (20)$$

A first-order approximation leads to

$$\delta\theta = J\delta\psi;$$

in other words, the changes on the AR parameters are constrained to the subspace range (J). The corresponding diagnosis test is nothing but (14) with

$$A = \mathcal{H}'_{p-1, N-1} J_{p, m}.$$

For example, if the diagnosis problem of interest is to monitor eigenfrequencies ω_j, the corresponding Jacobians may be found in [6]. The advantage of this approach is that it allows the separate monitoring of as many poles or subsets of poles as desired, without knowing a priori which poles will actually change. The main drawback is that a coupling effect may exist between the poles to be monitored; namely, all the separate tests can be nonzero even if only one pole has actually moved. However, it will be shown in Section IV that the diagnosis decision is nevertheless correct in most cases.

B. Decoupling Method

The basic idea of this approach is to reidentify the poles which have to be monitored and to use the global tests (15) or (18) associated to the small order corresponding U vectors. For simplicity, let us consider the case where no pole is real, and thus $p = 2r$.

Define

$$P(z) = z^{2r} - \sum_{i=1}^{2r} a_i z^{2r-i},$$

the characteristic polynomial of the model, and let us consider all the possible factorizations of the form

$$P(z) = (z - \lambda_j)(z - \bar{\lambda}_j) P_j(z).$$

The decoupling method for diagnosis is as follows. For each

index j of interest, achieve the inverse filtering of the signal (y_t) through P_j. On the resulting signal, identify the AR part of an ARMA $(2, q)$ model, in the same manner as in Section II-A for example. Then, using the new "nominal" values a_1^j and a_2^j, compute the corresponding U^j vector via (4) and the χ^2 test (18),

$$T^j = U^{j\prime} \hat{\Sigma}_j^{-1} U^j$$

with $p = 2$ and with Σ estimated via (9) or (19). If there is no change at all, all the T^j tests will be zero. If there is a change on the pole λ_i, then the test T^i is zero, while all the other ones $(T^j$ for $j \neq i)$ are nonzero.

The obvious advantage of this method is that there is no more coupling effect between the poles: if only one pole is moving, only one test is closed to the "good" value (here zero). The main drawback of this method is that if $m \geq 2$ poles are moving simultaneously, then we need to perform a number of tests that is equal to the number of combinations of m elements among n (order of the system). The sensitivity method requires only m tests. Furthermore, the decoupling method requires in practice the prior knowledge of which subsets of poles are moving, otherwise the decoupling property is lost.

Numerical results concerning these two methods for diagnosis will be given in the next section. We mention that their extension to the vector case is possible.

IV. NUMERICAL RESULTS

Here we investigate the numerical behavior of the tests which we have presented in the two previous sections. The main points to be emphasized are that

1) the global test (15) is an efficient approach, especially when the estimated Σ matrix is computed via (19);
2) the sensitivity method, despite its coupling effect, is able to detect and diagnose small changes in eigenfrequencies;
3) the decoupling method is efficient for diagnosis when only one pole moves.

The experiments which have been done are highly motivated by the fact that, in view of the application to vibration monitoring, we are interested in detecting small changes in eigenfrequencies, where small means one percent. In other words, according to the location of the corresponding poles, the "observable" change, namely the change in the cosine, may be less than four per thousand.

We have chosen models of even order, with pairs of complex conjugate poles, of the form

$$\left(\rho_j e^{i\omega_j}, \rho_j e^{-i\omega_j}\right),$$

and we have studied changes in one or more ω_j. In most cases, the ρ_j are equal, but the influence of these parameters has also been studied. We will show that a fixed pole close to the unit circle can prevent the diagnosis, and even the "global" detection of a change in a second pole far from the unit circle. The models which have been used are shown in Table I. For each experiment, the numerical values of the test statistics are computed under both H_0 (i.e., the actual model is the reference model) and H_1 (i.e., the actual model is the changed model).

Table II gives the values of the global test and the sensitivity test for diagnosis in the special case of stationary AR signals. The reason for considering this case is the analysis of the coupling effect during diagnosis mentioned in Section III-A.

Table III gives the corresponding results for the nonstationary ARMA $(p, p - 1)$ case, where the MA coefficients are piecewise constant. The lengths of the intervals are randomly chosen; the jumps in the MA coefficients have the same order of magnitude as the changes to be detected in the AR coefficients. No special attention was given to the problem of pole-zero cancellation (unlikely to occur).

TABLE I
NOMINAL AND CHANGED MODELS USED
FOR THE SIMULATION STUDY

Name of Experiment	Reference Model ρ_j	ω_j	Changed Model ω_j
AR4 $e41$	0.99	1.9	1.9
	0.99	0.5	0.495
$e42$	0.99	0.8	0.8
	0.99	0.6	0.594
$e43$	0.99	1.5	1.5
	0.99	0.5	0.495
$e44$	0.99	2.2	2.18
	0.99	2.4	2.4
$e45$	0.98	1.9	1.9
	0.99	0.5	0.495
$e46$	0.99	1.9	1.9
	0.98	0.5	0.495
$e47$	0.98	0.8	0.8
	0.99	0.6	0.594
$e48$	0.99	0.8	0.8
	0.98	0.6	0.594
$e49$	0.99	0.8	0.8
	0.6	0.6	0.594
AR6 $e61$	0.99	1.9	1.9
	0.99	0.6	0.6
	0.99	0.4	0.396
$e62$	0.99	1.5	1.5
	0.99	0.6	0.6
	0.99	0.4	0.396
$e63$	0.99	1.5	1.5
	0.99	0.6	0.594
	0.99	0.4	0.4
$e64$	0.99	0.8	0.8
	0.99	0.6	0.6
	0.99	0.4	0.396
$e65$	0.99	0.8	0.8
	0.99	0.6	0.594
	0.99	0.4	0.4

TABLE II
INVESTIGATION OF THE COUPLING EFFECT OF THE SENSITIVITY
METHOD IN THE STATIONARY AR CASE

Experiment	H_0 Global Test	H_0 Sensitivity Test	H_1 Global Test	H_1 Sensitivity Test
$e41$	3.78	0.93	15.14	0.85
		0.60		14.15
$e42$	3.32	1.09	17.99	1.24
		0.93		17.90
$e43$	2.95	0.90	13.68	1.57
		1.25		11.81
$e44$	4.42	1.37	175.69	189.32
		1.06		1.18
$e45$	3.44	0.99	14.12	0.87
		0.69		13.57
$e46$	3.93	0.65	9.05	1.44
		0.61		7.69
$e47$	4.39	1.13	18.21	1.27
		0.87		15.94
$e48$	3.15	0.61	10.33	1.30
		1.09		7.79
$e49$	4.52	1.13	4.12	0.90
		1.87		1.01
$e61$	4.99	0.35	13.23	1.27
		1.09		1.30
		0.66		7.55
$e62$	6.54	0.61	13.32	1.82
		1.36		1.09
		0.87		10.96
$e63$	6.54	0.61	22.35	0.83
		1.36		16.74
		0.87		1.29
$e64$	7.01	1.07	17.51	1.96
		1.39		1.14
		1.74		13.50
$e65$	7.01	1.07	22.10	1.74
		1.39		17.43
		1.74		1.68

TABLE III
DIAGNOSIS BY SENSITIVITY AND DECOUPLING METHODS
IN THE NONSTATIONARY ARMA CASE (PIECEWISE
CONSTANT MA COEFFICIENT)

Experiment	H_0 Global Test	H_0 Sensitivity Test	H_0 Decoupling Test	H_1 Global Test	H_1 Sensitivity Test	H_1 Decoupling Test
$e41$	2.56	0.89	2.39	14.90	0.43	15.15
		0.49	2.00		13.20	1.81
$e42$	1.75	0.48	0.75	11.29	0.84	22.74
		0.59	0.69		9.27	0.75
$e43$	2.57	1.08	1.97	7.23	0.29	17.63
		0.49	0.93		6.33	0.82
$e44$	4.29	0.70	7.55	237.35	233.03	6.22
		1.54	4.86		1.23	312.29
$e61$	2.27	1.17	4.14	9.84	0.44	13.24
		0.26	1.72		0.30	11.10
		0.61	1.84		6.70	3.80
$e62$	2.69	0.54	5.09	7.27	0.92	9.21
		0.30	6.10		0.26	7.44
		0.36	2.66		3.93	3.64
$e64$	4.11	0.17	5.53	8.99	0.84	20.60
		0.25	4.09		0.47	13.72
		0.62	6.20		4.21	3.63

It must be emphasized that the numerical values given in these tables are pessimistic because they give the mean values obtained by mixing experiments with different record lengths (from 1000 to 10 000, by increments of 500), and in many experiments it is obvious that the behavior of the tests (especially those for diagnosis) is very poor when less than 3000 or 4000 sample points are available (remember the small magnitude of the changes under study).

Let us finally comment upon the choice of the parameters of the algorithms, namely, the AR order p, the number of instruments N, and the thresholds. It can be shown that if the AR order p is overestimated, the law of large numbers (10) and the central limit theorem (11) and (12) are still valid; on the other hand, nothing can be said if this order is underestimated. The numerical results of Tables II and III were obtained when the algorithms were run with exact AR order; however, overestima-

tion of the order was also tried and gave good results, provided that the overestimated AR model did not add already-existing poles. Other experiments made on real (offshore platform) data show that it is possible to get interesting results, at least for the global test, with small orders ($p \simeq 10$, small for that application). As far as the number N of instruments is concerned, the present experiments have all been done with $N = p$; $N > p$ does not seem to improve the results in practice (when $\tilde{\Sigma}_N$ is used). However, it can be shown that the theoretical optimal number of instruments for the global test (15) is infinite; this issue is currently under investigation, and results are similar to those obtained by Stoica et al. in [14] for the identification problem. Finally, Tables II and III show that it is possible to (empirically) choose a threshold which discriminates between null and alternative hypotheses. The global test t_0 (15) is theoretically distributed as an χ^2 with p degrees of freedom, thus with mean value p. In practice, for simulated data, the mean value of t_0 is of comparable order of magnitude (possibly larger because we use $\tilde{\Sigma}$ (19) instead of $\hat{\Sigma}$ (9)); for real data, the thresholds are basically relative, and not absolute, partly because of the underestimation of the AR order.

V. CONCLUSION

The problem of detection and diagnosis of changes in modal characteristics of nonstationary (scalar) digital signals has been addressed. An equivalent problem is to detect changes in the poles of an ARMA model having nonstationary unknown moving-average coefficients. New tests have been derived and studied via a simulation study. The main idea underlying our approach is to use a likelihood ratio technique, but based upon an instrumental statistic (rather than the observations themselves) which is more robust with respect to the nuisance parameters. The main conclusion is that detection and diagnosis of small (one percent) changes in eigenfrequencies are possible. provided that the size of the sample is large enough (several thousands) and that there is no "masking effect," namely that the poles to be monitored are not less close to the unit circle than other ones. This latter point is to be investigated further. Finally, the extension of these tests to the case of vector signals [2] may be used as a solution to the problem of vibration monitoring and will be reported later.

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An Algorithm for Solving the Extended Yule–Walker Equations of an Autoregressive Moving-Average Time Series

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Abstract—A new form of the extended Yule–Walker equations of a stationary autoregressive moving-average (ARMA) scheme is proposed. An algorithm using the new form is also given for calculating the parameters of the ARMA process from its autocovariance function without a proof of its convergence.

I. INTRODUCTION

A stationary autoregressive (AR) process $\{x_t\}$ of order p can be fully identified from the first $p + 1$ autocovariances, that is $\text{cov}(x_t, x_{t+k})$, $k = 0, 1, \cdots, p$, by the Yule–Walker equations. Moreover, the Yule–Walker equations have been employed in estimating the AR parameters and the disturbance variance from the first $p + 1$ sample autocovariances by means of the Levinson–Durbin algorithm [1], [2] or Pagano's algorithm [3]. However, since the extended Yule–Walker equations of a stationary autoregressive moving-average (ARMA) process $\{x_t\}$ of order (p, q), $q > 0$, contain the cross covariance function between the process $\{x_t\}$ and the white noise process [4, p. 75], it is hard to calculate p AR parameters, q moving-average (MA) parameters, and the disturbance variance from the first $p + q + 1$ autocovariances, that is $\text{cov}(x_t, x_{t+k})$, $k = 0, 1, \cdots, p + q$, by dint of the extended Yule–Walker equations. Additionally, any method to estimate the parameters of the ARMA model, particularly the MA parameters, from the first $p + q + 1$ sample autocovariances through the extended Yule–Walker equations has not been used yet [5]. In this correspondence we derive another form of the extended Yule–Walker equations of the ARMA model, which does not contain the cross autocovariance function. This new form of the extended Yule–Walker equations shows the explicit relation between the $p + q + 1$ parameters and the first $p + q + 1$ autocovariances. Finally, we propose a simple iterative algorithm to solve the extended Yule–Walker equations; the algorithms convergence is an object of further study.