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Detection and Diagnosis of Changes in the Eigenstructure of Nonstationary Multivariable Systems*†

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Even when the state noise of a multivariable system is unknown and nonstationary, it is possible to monitor changes in its state transition matrix with the aid of new statistical tests closely related to the identification method suitable in such a situation

Key Words—Time-varying multivariable systems, failure detection, diagnosis, vibration monitoring

Abstract—The two problems of detection and diagnosis of changes in the state transition matrix of a multivariable system with nonstationary unknown state noise are addressed New instrumental tests are derived and shown to be numerically powerful, even for small changes The application to vibration monitoring of offshore platforms is described

1 INTRODUCTION

THE PROBLEM OF detecting changes in dynamical systems has received a lot of attention (Willsky, 1976, Mironovski, 1980, Isermann, 1984, Basseville and Benveniste, 1986) in many fields of application, for the purpose of failure detection in controlled systems or signal segmentation for recognition Most of the time domain model-based methods use the entire set of known or estimated model parameters for solving the two basic steps of change detection, namely residual generation and choice of the (statistical) decision function (Willsky, 1976) For example, both filter innovations and parity checks involve all the model parameters, with possible inclusion of parameter uncertainties, and classical likelihood ratio or bayesian tests proceed similarly

However, for some applications it may be necessary to detect changes in one subset of the model parameters without knowing or using any estimates of the parameters belonging to the complementary subset These last parameters therefore have to be considered as nuisance parameters In Section 4, such an application concerned with vibration monitoring of a structure subject to nonstationary and unmeasured natural excitation will be described The related change detection problem may be formulated in the two following equivalent ways

(i) Consider a multivariable observable system described by the discrete time state space model

$$\begin{cases} X_{t+1} = FX_t + V_{t+1}, & \operatorname{cov}(V_{t+1}) = Q_t \\ Y_t = HX_t \end{cases} \tag{1}$$

where the dimensions of the state X and the observation Y are n and r, respectively, with r (much) smaller than n in practice, and where the additive state noise V_t is an *unmeasured* Gaussian process with *time-varying* covariance matrix Q_t Then the problem is to detect changes in the state transition matrix F, up to a change of basis, without using the nuisance parameters Q_t which may be highly time-varying (see Section 4) Here the observation matrix H is assumed to be fixed, the related problem of optimal sensor location for change detection is reported in Basseville *et al* (1986b)

(ii) Consider a multidimensional ARMA process

$$Y_t = \sum_{i=1}^{p} A_i Y_{t-i} + \sum_{j=0}^{p-1} B_j(t) E_{t-j} \tag{2}$$

with *constant* autoregressive $r \times r$ parameters $(A_i)_{1 \leq i \leq p}$ and *time-varying* moving average $r \times r$ parameters $(B_j)_{0 \leq j \leq p-1}$, and where $(E_t)_t$ is a Gaussian white noise with identity covariance matrix The model (2) may be obtained from (1) in a classical way (Akaike, 1974) by solving the following

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linear system of equations

$$HF^p = \sum_{i=1}^{p} A_i H F^{p-i} \quad (3)$$

Let

$$\theta^T = (A_p, \quad , A_1) \quad (4)$$

be the set of AR parameters

In this situation, the problem is to detect changes in the AR parameters θ without using the nuisance moving-average parameters $B_j(t)$ $(0 \leqslant j \leqslant p-1)$

In each of these two model formulations, the second problem addressed here is the problem of diagnosis when a change has been detected, one must decide which pole(s) and eigenvector(s) have changed In (1), it is of interest to monitor the eigenstructure of the system, i e the pairs $(\lambda, H\phi_\lambda)$ where λ and ϕ_λ are the eigenvalues and corresponding eigenvectors of the state transition matrix F, which is assumed to be asymptotically stable In (2), equivalently, the generalized eigenvalues and eigenvectors of the matrix polynomial

$$I - \sum_{i=1}^{p} A_i z^{-i}$$

will be considered

In Section 4, these will be the vibrating characteristics of the monitored structure

Because of the timing characteristics of the authors' application, namely very slowly occurring changes (months or years) and high sampling frequency, in all what follows a model validation problem rather than a change detection problem will be solved namely, given a reference model F^0 or θ^0, and a new record of observations $(Y_t)_{1 \leqslant t \leqslant s}$, decide whether this model still adequately describes it (global detection) and solve the diagnosis problem However, the tests which are proposed here may be used for on-line change detection purposes, as generally explained in Basseville and Benveniste (1986) this has been done for segmentation of speech signals in André-Obrecht (1986)

The paper is organized as follows In Section 2 the authors' original approach for solving the global detection problem will be presented, using both an instrumental statistic and a statistical local approach for detection In Section 3, the diagnosis problem will be studied, following the same local approach and using a linearization to relate changes in eigencharacteristics to changes in AR parameters Numerical experiments on both simulated and real data are reported in Section 5, after the description of the underlying mechanical engineering problem of vibration monitoring, which is given in Section 4 Finally, some conclusions are presented in Section 6

2 GLOBAL CHANGE DETECTION

In this section the two equivalent problems of detecting changes in the F matrix of model (1) or in the AR parameters $(A_i)_i$ of model (2) are investigated, with as much robustness as possible with respect to the non-stationary unknown excitation V_t or moving average parameters $(B_j(t))_j$

As discussed in Basseville et al (1986a) in the scalar case, it is not possible to use standard likelihood techniques for solving this problem The two key reasons are that, first, the Fisher information matrix of an ARMA process is not block-diagonal, and thus there is a coupling effect between poles and zeros, and, second, the highly time-varying behavior of the unknown parameters prevents one using nuisance parameter elimination methods, usually based upon estimation or integration Recall that in Bohlin (1977), for example, convenient estimated values of the MA coefficients were used

The idea of the authors' approach is to transform the problems (i) or (ii) into the simpler problem of detecting a change in the mean of a Gaussian process with known covariance matrix, which is then solved by the classical χ^2 (generalized) likelihood ratio test

2 1 *An instrumental statistic*

For the above mentioned purpose, consider what we call the instrumental statistics

$$U_N(s) = \sum_{t=1}^{s} Z_t W_t^{\mathrm{T}} \quad (5)$$

where

$$Z_t^{\mathrm{T}} = (Y_{t-p}^{\mathrm{T}}, \quad , Y_{t-p-N+1}^{\mathrm{T}})$$

is a vector of $N \geqslant p$ instrumental variables, i e here simply delayed observations, and where W_t is

$$W_t = Y_t - \sum_{i=1}^{p} A_i^0 Y_{t-i} = Y_t - \theta^{0\mathrm{T}} \phi_t \quad (6)$$

with

$$\phi_t^{\mathrm{T}} = (Y_{t-p}^{\mathrm{T}}, \quad , Y_{t-1}^{\mathrm{T}})$$

θ^0 is the reference AR model, which has to be validated on the new record $(Y_t)_{1 \leqslant t \leqslant s}$ The process $U_N(s)$ may be alternatively generated in a practically more attractive way, using the following formula

$$U_N(s) = \mathscr{H}_{p+1,N}^{\mathrm{T}}(s) \begin{pmatrix} \theta^0 \\ -I_r \end{pmatrix} \quad (7)$$

where $\mathscr{H}_{p+1,N}(s)$ is the empirical Hankel matrix of the observed process (Y_t)

$$\mathscr{H}_{p,q}(s) = \begin{pmatrix} R_0(s) & & R_{q-1}(s) \\ & & \\ R_{p-1}(s) & & R_{p+q-2}(s) \end{pmatrix},$$

$$R_m(s) = \sum_{t=1}^{s-m} Y_{t+m} Y_t^{\mathrm{T}}$$

It is well known (Soderstrom and Stoica, 1980) that equating the right-hand side of (7) to zero and solving, in the least-squares sense if $N > p$, the resulting system of delayed Yule–Walker equations, is nothing but the *instrumental variable* I V identification method, the consistency of which is established under stationary conditions Benveniste and Fuchs (1985) show that the I V method also leads to a *consistent* estimate of the AR parameters θ, in the present situation of *nonstationary unknown* MA coefficients This last robustness property is numerically proved in Prevosto *et al* (1982) and in Prevosto *et al* (1983) where favorable comparisons with frequency domain methods are made The detection problem for the process U_N is now investigated

Under the hypothesis H_0 of no change, i e $\theta = \theta^0$, W_t defined by (6) is actually a MA process, which is uncorrelated with Z_t, and thus $U_N(s)$ is zero-mean

Under the local alternative hypothesis H_1 of small change, i e $\theta = \theta^0 + \delta\theta/\sqrt{s}$, the mean of $U_N(s)$ can be easily checked to be equal to the mean of

$$\frac{1}{\sqrt{s}} \mathscr{H}_{p\,N}^{\mathrm{T}}(s)\,\delta\theta \qquad (8)$$

In order to be able to compute covariances, it is necessary to consider a convenient vector $\mathscr{U}_N$ instead of the matrix U_N, and from now on the notations and basic results concerning Kronecker products which are summarized for example in Yuan and Ljung (1984) will be used

Let

$$\begin{aligned} \mathscr{U}_N(s) &\triangleq \operatorname{col}(U_N^{\mathrm{T}}(s)) \\ &= \sum_{t=1}^{s} Z_t \otimes W_t \end{aligned} \qquad (9)$$

be the vector of size Nr^2 obtained by stacking the Nr columns of $U_N^{\mathrm{T}}(s)$ on top of each other, and $\Theta = \operatorname{col}(\theta^{\mathrm{T}})$ be obtained in the same way from θ (4) Furthermore, let $\Sigma_N(s)$ be the covariance matrix of $\mathscr{U}_N(s)$ under the no change hypothesis H_0 Then, because of the independance of (Z_t, W_t) and (Z_s, W_s) for $|t-s| \geqslant p$

$$\Sigma_N(s) = \sum_{t=1}^{s} \sum_{i=-p+1}^{p-1} \mathscr{E}_0(Z_t Z_{t-i}^{\mathrm{T}} \otimes W_t W_{t-i}^{\mathrm{T}}), \qquad (10a)$$

where $\mathscr{E}_0$ is the expectation under hypothesis H_0

Consider the following estimate

$$\hat{\Sigma}_N(s) = \sum_{t=1}^{s} \sum_{i=-p+1}^{p-1} (Z_t Z_{t-i}^{\mathrm{T}} \otimes W_t W_{t-i}^{\mathrm{T}}) \qquad (10b)$$

It is shown in Moustakides and Benveniste (1986) that $(1/s)\hat{\Sigma}_N(s)$ is a *consistent* estimate of $(1/s)\Sigma_N(s)$ under *both* hypotheses H_0 and H_1 (because first order approximations do not affect covariances), and that $(1/\sqrt{s})\mathscr{U}_N(s)$ is asymptotically *Gaussian* distributed under both hypotheses, i e

$$\Sigma_N^{-1/2}(s)\mathscr{U}_N(s) \underset{s\to\infty}{\overset{\mathscr{L}}{\to}} \mathscr{N}(0, I) \quad \text{under } H_0$$

$$\Sigma_N^{-1/2}(s)(\mathscr{U}_N(s) - \mu(s)) \underset{s\to\infty}{\overset{\mathscr{L}}{\to}} \mathscr{N}(0, I) \quad \text{under } H_1$$

where

$$\mu(s) = \frac{1}{\sqrt{s}}(\mathscr{H}_{p,N}^{\mathrm{T}}(s) \otimes I_r)\,\delta\Theta \qquad (11)$$

It is emphasized that these law of large numbers and central limit theorems are obtained in the *nonstationary* framework which has been assumed here (time-varying moving average parameters) Therefore, as stated at the beginning of this section, the initial problems (i) and (ii) have been transformed into the classical problem of change in the mean of a Gaussian process with known covariance matrix Precise assumptions and theorem statements are given in Appendix A

2 2 *The changes which can be detected*

The changes which can be detected with the aid of the instrumental statistics $U_N(s)$ (7) are now emphasized Because of the mean value (8) of $U_N(s)$ under H_1, none of the changes $\delta\theta$ belonging to the kernel of $(1/s)\mathscr{H}_{p,N}^{\mathrm{T}}(s)$ will be detected

It is assumed that the nominal representation (H_0, F_0) (1) is observable and that the following factorization of the empirical Hankel matrix holds

$$\frac{1}{s}\mathscr{H}_{p,N}(s) = \mathscr{O}_p(H_0, F_0)\; \mathscr{C}_N(F_0, G_s) + \varepsilon(s) \quad (12)$$

where

$$\mathscr{O}_p(H_0, F_0) = \begin{pmatrix} H_0 \\ H_0 F_0 \\ \\ H_0 F_0^{p-1} \end{pmatrix}$$

is the observability matrix, and

$$\mathscr{C}_N(F_0, G_s) = (G_s, F_0 G_s, \quad , F_0^{N-1} G_s)$$

is the controllability matrix, uniformly of full row rank n, G_s being the empirical cross-correlation between the state X and the observation Y In (12), $\varepsilon(s)$ converges to zero in distribution as $s \to \infty$ Refer to Appendix A for the assumptions under which these properties are true

In such a case, because of the factorization (12), the only changes on θ which will not be able to be detected with this approach are those that satisfy

$$\mathcal{O}_p^{\mathrm{T}}(H_0, F_0)\,\delta\theta = 0 \qquad (13)$$

But these last changes do *not* correspond to any change in the minimal representation (1) of the system, as the following argument shows In fact, (3) shows that the representations (1) and (2) of the system are related to each other through the relationship

$$\mathcal{O}_{p+1}^{\mathrm{T}}(H_0, F_0)\begin{pmatrix} \theta_0 \\ -I_r \end{pmatrix} = 0$$

and any θ_0 satisfying this relation gives rise to a valid ARMA representation of the system But two different parameters θ_0 and $\theta_0 + \delta\theta$ satisfying the above relation are precisely related through (13) Further discussions may be found in Benveniste and Fuchs (1985) This property (13) is of particular importance in practice, because it means that, for detecting changes in a minimal representation (H_0, F_0) (1) of (Y_t) with the aid of the instrumental statistics $U_N(s)$ (7), it is not necessary to use a minimal ARMA representation (2) of (Y_t)

2 3 *The global test*

The test for detecting changes in the mean of $\mathscr{U}_N(s)$ (9) is now given Recall that, if a vector U is Gaussian with mean μ and covariance Σ, for testing $\mu = 0$ against $\mu \in \mathrm{Range}(M)$ where M is a full column rank matrix, the (generalized) likelihood ratio test is

$$U^{\mathrm{T}}\Sigma^{-1}M(M^{\mathrm{T}}\Sigma^{-1}M)^{-1}M^{\mathrm{T}}\Sigma^{-1}U \qquad (14)$$

Consider the application of this result to the instrumental statistics $\mathscr{U}_N(s)$

In order to apply (14), it is necessary to reduce

$$M = \mathscr{H}_{p\,N}^{\mathrm{T}}(s) \otimes I_r \qquad (15)$$

to a full (nr) column rank matrix This is possible because the system (1) is assumed to be observable A solution involving a selection of rows in $\mathcal{O}_p(H_0, F_0)$ and of columns in $\mathscr{H}_{p\,N}(s)$ is described in Basseville *et al* (1985) and Rougee (1985)

Therefore, according to the previous discussion, the global test for detecting changes in Θ is

$$t_0 = \mathscr{U}_N^{\mathrm{T}}(s)\hat{\textstyle\sum}_N^{-1}(s)M \times \left(M^{\mathrm{T}}\hat{\textstyle\sum}_N^{-1}(s)M\right)^{-1} M^{\mathrm{T}}\hat{\textstyle\sum}_N^{-1}(s)\mathscr{U}_N(s) \qquad (16)$$

with M given by (15) Under H_0, t_0 is distributed as a χ^2 variable with nr degrees of freedom Under H_1, t_0 is distributed as a χ'^2 with the same number of degrees of freedom, and with noncentrality parameter

$$\delta\Theta^{\mathrm{T}}M^{\mathrm{T}}\hat{\textstyle\sum}_N^{-1}(s)M\,\delta\Theta \qquad (17)$$

which is non-zero in all the cases of interest as previously discussed This χ^2 behavior may be used for determining a threshold for detection, provided that the true model of the monitored system has an AR part with the same order as the order used for the instrumental test For many real systems, because of the underestimation of the AR order, the threshold to be used is basically relative, and not absolute (Basseville *et al*, 1986a)

An interesting special case will briefly be mentioned which will be useful in practice Assume that the number of sensors r divides the state dimension n Then generically [if the first n rows of $\mathcal{C}_p(H_0, F_0)$ are independent], the AR order of the ARMA model (2) is $p = n/r$ On the other hand, the minimum number of instruments to be used is $N = p$ (Benveniste and Fuchs, 1985) In this situation, M (15) is invertible and the global test reduces to

$$t_0 = \mathscr{U}_p^{\mathrm{T}}(s)\hat{\textstyle\sum}_p^{-1}(s)\mathscr{U}_p(s) \qquad (18)$$

The efficiency of the test (16) is numerically investigated in Section 5, and theoretically analyzed by Rougée (1985) and Rougée *et al* (1985) under stationarity assumptions It turns out that, in the case where $n = pr$, i e where the dimension of the observation is a divisor of the state dimension, the asymptotic power of the test t_0 is related to the asymptotic precision of the instrumental variable identification method, as derived by Stoica *et al* (1985) Furthermore, it may be shown that, in the scalar case and for a special choice of instruments, this test is equivalent to the min–max optimal local likelihood test, which is *robust* with respect to uncertainties on the moving average part Finally, preliminary numerical results for nonstationary scalar signals are described by Basseville *et al* (1986a) and show that small changes (1%) in eigenfrequencies may be detected, using sufficiently long records

3 THE DIAGNOSIS PROBLEM

The diagnosis problem is now discussed, as stated in the Introduction When a change in the AR part has been detected by the global test t_0, the problem is to get insights into which poles and modes have changed, again without knowing the nonstationary MA part As far as the authors know, the multiple model approach is the only solution which has ever been given to the diagnosis problem in general, as discussed in Willsky (1976) and Willsky (1986)

The authors' suggestion is to use the same approach as for the global test, together with a relationship between changes in eigen-characteristics and changes in AR parameters, to be used in the expression (11) of the mean value of the instrumental statistics $\mathcal{U}_N$ under H_1 As the authors' detection tools are developed under the assumption of small changes, first order Taylor expansions for Θ are used in the following manner

Let Φ be a minimal parameterization of the AR part of the process In Section 4, Φ will be the set of the eigen (or modal) characteristics of the structure Assume that the application $\Phi \rightarrow \Theta = f(\Phi)$ is continuously differentiable in the neighborhood of the nominal model Φ_0 To monitor a particular subset of the coordinates of Φ, consider the matrix $\mathscr{J}$ obtained by selecting the corresponding columns of the Jacobian matrix $f'(\Phi_0)$, and apply formula (16) with

$$M = (\mathscr{H}^{\mathrm{T}}_{p,N}(s) \otimes I_r)\mathscr{J} \tag{19}$$

in order to get what we call a *sensitivity test*

Several computations of Jacobians of interest for the vibration monitoring application are described in Section 4

The main advantage of this method is that it allows separate monitoring of subsets of parameters of interest (for example, one pole together with the corresponding eigenvector), without knowing in advance which subsets will actually change The main drawback is that no theoretical argument may assess some decoupling property, concerning for example separate monitoring of modes the corresponding tests (16) computed with the relevant Jacobians (19) are not statistically independent of each other However, simulation results presented by Basseville *et al* (1986a) show that, in the scalar case ($r = 1$), this decoupling property concerning poles holds, provided that the poles are close to unit circle In the vector case, this seems to be still true, provided that furthermore the number of sensors r is equal to the actual number of modes ($n/2$ in this case of real F), see Basseville (1985)

Finally, another approach for the diagnosis problem which is investigated by Basseville *et al* (1986a) in the scalar case will be mentioned This approach is based upon a theoretical decoupling property, but is difficult to implement because of combinatorial problems in the (real) case of no *a priori* knowledge of which parameter subsets are to change

4 APPLICATION TO THE VIBRATION MONITORING PROBLEM

4 1 *Motivations*

As mentioned in the Introduction, the reason for which the problems addressed in this paper were studied is a mechanical engineering problem how to supervise the vibrating characteristics of a structure subject to a nonstationary and unmeasured natural excitation Examples of such vibrating structures are offshore platforms subject to the swell (the purpose of the authors' study), buildings or bridges subject to wind or earthquakes, mechanical objects subject to fluid interactions, etc One of the goals of vibration monitoring is the detection of cracks and fatigue The authors' approach for solving such a problem involves the following steps

(i) *On site identification of the vibrating characteristics of the structure* This step is necessary because the finite element models provided by the designer have to be significantly adjusted when the structure is installed in the sea Furthermore, the possible model deviations due to fatigue appearance are often less important than the deviations between the designer's model and the behavior actually observed when the structure is installed in the sea Therefore the designer's model cannot be used as a reference model for fatigue detection by modal analysis

(ii) *Detection of changes in the modal characteristics* As explained in the Introduction, the purpose of this task is to decide whether the model which was identified on the safe structure still adequately represents the new accelerometer signals obtained during a new inspection

(iii) *Diagnosis of the change* The problem is then to discriminate between changes in the distribution of the masses which are of no interest for fatigue detection, and changes in the stiffness coefficients, and furthermore to estimate the localization of the fatigues in the structure

From the implementation point of view, tasks (i) and (iii) should be done in a remote computing center because they are time consuming Task (ii), previously called global detection in Section 2, should be done on board

To the authors' knowledge, it seems that no other parametric model approach has been followed for solving these types of problems Furthermore, the way in which the authors deal with the excitation and the fluid/structure interactions also seems to be nonstandard Because the vibrating modes of an offshore platform lie beyond 1 Hz (and thus beyond the fundamental frequency of the swell), the structure is mainly excited by shock effects and turbulences which induce nonstationarities in the signals and which are *unmeasurable* by the existing swell sensors In such a situation, the authors' approach is to model the excitation by a nonstationary Gaussian white noise, to neglect the fluid/structure interactions, and to develop algorithms as robust as possible with respect to these assumptions, for the three above-mentioned tasks

4 2 *The model to be used*

Assuming that the structure may be decomposed into finite elements and has a linear behavior (Prevosto, 1982), a vibrating structure is modelled by a damped system of masses connected by springs, which obeys the following equation

$$MD_t + CD_t + KD_t = \varepsilon_t \tag{20}$$

where M, C, K are the masses, damping and stiffness matrices, respectively, D_t is the vector of displacement of the m degrees of freedom, and ε_t is the excitation vector, assumed to be a nonstationary Gaussian white noise with covariance matrix R_t The modal characteristics (λ, ϕ_λ) of this structure are the solutions of

$$\begin{cases} \det(M\lambda^2 + C\lambda + K) = 0 \\ (M\lambda^2 + C\lambda + K)\phi_\lambda = 0 \end{cases} \tag{21}$$

As M, C, K are real, the $2m$ solutions $\lambda = c + i\omega$ are pairwise conjugate They are called eigenfrequencies, the ω are the eigen-pulsations, and $c/\sqrt{c^2 + \omega^2}$ is the corresponding (negative) damping coefficient In case of proportional damping, i e when $C = \alpha M + \beta K$, the modes ϕ_λ are real

Let Y_t be the observation of $r \leqslant m$ degrees of freedom of the system Y_t is described by

$$Y_t = LD_t \tag{22}$$

where L_{ij} is equal to one if sensor i observes the degree j, and zero otherwise Then the modal identification is the obtainment of $(\Delta, L\Phi)$ where Δ is the diagonal matrix filled by the λ, and $L\Phi$ is the observed part of the modes ϕ_λ

The system described by (20) and (22) is equivalent to the following continuous time state space model

$$\begin{cases} X_t = AX_t + B_t \\ Y_t = HX_t \end{cases} \tag{23}$$

where

$$X_t = \begin{pmatrix} D_t \\ D_t \end{pmatrix}, \quad \dim X_t = 2m = n,$$

$$A = \begin{pmatrix} 0 & I \\ -M^{-1}K & -M^{-1}C \end{pmatrix},$$

$$B_t = \begin{pmatrix} 0 \\ M^{-1}\varepsilon_t \end{pmatrix},$$

$H = (L \quad 0)$ The discrete time equivalent model is

$$\begin{cases} X_{t+\Delta t} = FX_t + V_t \\ Y_t \ = HX_t \end{cases} \tag{24}$$

where $F = e^{A\,\Delta t}$ and

$$V_t = \int_t^{t+\Delta t} e^{A(t+\Delta t-\tau)} B_\tau \, d\tau$$

Because of the above assumption on ε_t (20), V_t is a white noise with covariance matrix

$$Q_t = \int_t^{t+\Delta t} e^{A\tau} \, \tilde{R}_\tau \, e^{A^T\tau} d\tau$$

where $\tilde{R}_\tau = \begin{pmatrix} 0 & 0 \\ 0 & M^{-1}R_\tau M^{-1} \end{pmatrix}$

The eigenvalues μ and eigenvectors ψ_μ of F(24) are related to the modal characteristics (21) by

$$\mu = e^{\lambda\,\Delta t} \quad \text{and} \quad H\psi_\mu = L\phi_\lambda \tag{25}$$

and are solutions of

$$\left(\mu^p I_r - \sum_{j=1}^{p} \mu^{p-j} A_j\right) H\psi_\mu = 0 \tag{26}$$

where the autoregressive parameters $A_1, \ \ , A_p$ are given by (3) Actually, assuming that $\Delta t = 1$, F may be diagonalized in the following manner

$$F = \Psi \, e^D \, \Psi^{-1} \tag{27}$$

where

$$D = \begin{pmatrix} \Delta & 0 \\ 0 & \overline{\Delta} \end{pmatrix} \quad \text{and} \quad \Psi = \begin{pmatrix} \Phi & \overline{\Phi} \\ \Phi\Delta & \overline{\Phi\Delta} \end{pmatrix}$$

Consequently, identifying and monitoring the set of $(\mu, H\psi_\mu)$ given by (26) is equivalent to the same tasks for the set of $(\lambda, L\phi_\lambda)$ given by (21) The relevance of the problems and methods investigated in the previous sections is thus stated for the vibration monitoring problem

Before describing the experiments which have been done for validating the authors' approach for the detection and diagnosis problems, recall that, for the identification problem (i) using (3) and (26), strong results have been obtained from both theoretical and experimental points of view Several known identification methods (instrumental variable, balanced realization, Ho-Kalman) have been shown to lead to consistent estimates $\hat{A}_i$ in (3), even with a nonstationary excitation (fast changes in the geometry of the excitation are allowed), see Benveniste and Fuchs (1985) From the experimental point of view, Prevosto *et al* (1983) have shown that these methods are more convenient than classical Fourier methods for the identification of high order modes, and they investigated the stability of the identified modes with respect to the waves, height and the swell direction Other numerical results for the identification problem may be found in Prevosto (1982) and Prevosto *et al* (1982)

4 3 *The diagnoses to be investigated*

With respect to the diagnosis problem statement (iii) which was given in paragraph 4 1, the purpose here is more restrictive The diagnoses to be considered are only in terms of the vibrating characteristics $(\lambda, L\phi_\lambda)$, rather than in terms of the physical parameters M, C, K This last point is currently under study, and has not been addressed before because it is a (highly) ill-conditioned problem in practice, the number of sensors r is between two and five, and the size of the M, C, K matrices is several hundred Notice that Isermann (1985) also assumes for diagnosis that the process coefficients of interest may be uniquely determined from the model parameters at hand Therefore, only the problems described in Section 3 will be considered, and results concerning some Jacobians (19) of interest for monitoring the vibrating or modal characteristics $(\Delta, L\Phi)$ (21) will be presented The key point is the computation of convenient differentiations of (3), which may be written equivalently,

$$(\theta^{T} \quad -I_r)\,\mathcal{O}_{p+1}(H,F) = \sum_{j=0}^{p} A_j H F^{p-j} = 0 \qquad (28)$$

where $A_0 = -I_r$ After the change of basis defined by (27), the following holds

$$(\theta^{T} \quad -I_r)\,\mathcal{O}_{p+1}(L\Phi, e^{\Delta}) = \sum_{j=0}^{p} A_j L\Phi\, e^{(p-j)\Delta} = 0 \qquad (29)$$

where

$$\mathcal{O}_p(L\Phi, e^{\Delta}) = \begin{pmatrix} L\Phi \\ L\Phi e^{\Delta} \\ \\ L\Phi e^{(p-1)\Delta} \end{pmatrix}$$

Then, differentiating (29) results in

$$\begin{aligned} d\theta^{T}\ \mathcal{O}_p(L\Phi, e^{\Delta}) &= -\sum_{j=0}^{p} A_j L\, d(\Phi\, e^{(p-j)\Delta}) \\ &= -\sum_{j=0}^{p} A_j L\, d\Phi\, e^{(p-j)\Delta} \\ &\quad -\sum_{j=0}^{p-1} (p-j)A_j\ L\Phi\ e^{(p-j)\Delta}\ d\Delta \end{aligned}$$

The corresponding equation for $d\Theta$, where $\Theta = \mathrm{col}(\theta^{T})$ was introduced in Section 2 1, is then

$$\begin{aligned} &(\mathcal{O}_p(L\Phi, e^{\Delta})^{T} \otimes I_r)\, d\Theta \\ &\quad = -\sum_{j=0}^{p} (e^{(p-j)\Delta} \otimes A_j L)\, \mathrm{col}\,(d\Phi) \\ &\quad -\sum_{j=0}^{p-1} (I_m \otimes (p-j)A_j L\Phi\, e^{(p-j)\Delta})\, \mathrm{col}\,(d\Delta) \end{aligned} \qquad (30)$$

After some computations which may be found in Appendix B,

$$(\mathcal{O}_p^{*T} \otimes I_r)\, d\Theta = \begin{pmatrix} J_\psi & 0 \\ 0 & \bar{J}_\psi \end{pmatrix}\begin{pmatrix} d\psi \\ \overline{d\psi} \end{pmatrix} + \begin{pmatrix} J_\lambda & 0 \\ 0 & \bar{J}_\lambda \end{pmatrix}\begin{pmatrix} d\lambda \\ \overline{d\lambda} \end{pmatrix} \qquad (31)$$

where $\mathcal{O}_p^{*}$ is the observability matrix in the modal basis and is easily shown to be

$$\mathcal{O}_p^{*} = (\mathcal{O}_p(L\Phi, e^{\Delta}), \overline{\mathcal{O}_p(L\Phi, e^{\Delta})}),$$

where

$$d\lambda = \begin{pmatrix} d\lambda_1 \\ \\ d\lambda_m \end{pmatrix}, \quad d\psi = \begin{pmatrix} L\, d\phi_1 \\ \\ L\, d\phi_m \end{pmatrix}$$

contain the variations of the observed part of the modes,

$$J_\psi = \begin{pmatrix} A(\mu_1) & 0 \\ \\ 0 & A(\mu_m) \end{pmatrix},$$

$$J_\lambda = \begin{pmatrix} \mu_1 A'(\mu_1) L\phi_1 & 0 \\ \\ 0 & \mu_m A'(\mu_m) L\phi_m \end{pmatrix}$$

with

$$\mu_j = e^{\lambda_j} \quad (1 \leqslant j \leqslant m),$$

and

$$A(X) = -\sum_{j=0}^{p} A_j X^{p-j}$$

Use of the following

$$d\psi = d\psi^{re} + i\, d\psi^{im}$$

and

$$d\lambda_j = dc_j + i\, d\omega_j \quad \text{[see (21)]}$$

gives

$$(\mathcal{O}_p^* \otimes I_r)\, d\Theta = \begin{pmatrix} J_\psi & iJ_\psi \\ \overline{J_\psi} & -i\overline{J_\psi} \end{pmatrix} \begin{pmatrix} d\psi^{re} \\ d\psi^{im} \end{pmatrix} + \begin{pmatrix} J_\lambda \\ \overline{J_\lambda} \end{pmatrix} dc + \begin{pmatrix} iJ_\lambda \\ -i\overline{J_\lambda} \end{pmatrix} d\omega \qquad (32)$$

Various vectors β containing any free modal parameters among the Ψ_j, c_j, ω_j $(1 \leqslant j \leqslant m)$ may then be monitored, by selecting the corresponding columns in the right-hand side of (32), in order to get a full column rank matrix J_β such that

$$(\mathcal{O}_p^* \otimes I_r)\, d\Theta = J_\beta\, d\beta \qquad (33)$$

Using the factorization (12) written in the modal basis, it turns out that, because of (19) and (33), the sensitivity test for monitoring β is given by (16) with

$$M = (\mathscr{C}_N^{*T} \otimes I_r)\, J_\beta \qquad (34)$$

where $\mathscr{C}_N^* = \Psi^{-1}\mathscr{C}_N(F, G_s)$

In the experiments which will be described in the next section, the selected vectors β are separate vibrating modes together with the corresponding vibrating pulsation, namely

$$\psi_j^{re}, \psi_j^{im}, \omega_j$$

The damping coefficients c_j are not monitored because they are usually not precisely identified

Finally, it will be mentioned that, as for the global test t_0 (16), theoretical investigations have been made by Rougée (1985) for computing the power of any sensitivity test of the form (16) with M given by (34) These results are also discussed in Rougée *et al* (1985) and are used in Basseville *et al* (1986b)

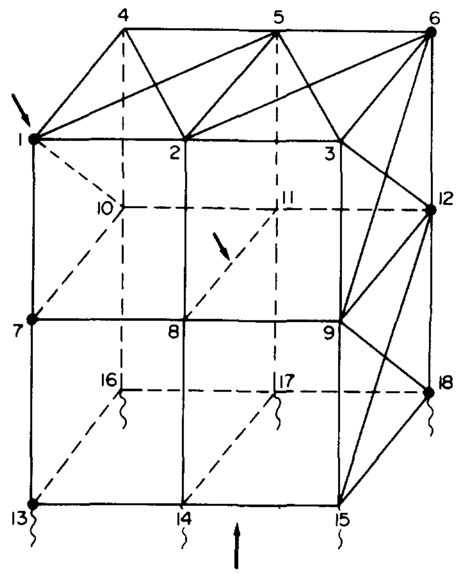

FIG 1 Simulation system

5 EXPERIMENTAL RESULTS

In this section, results obtained from the computer implementation of the various tests presented in Sections 2 and 3 are presented, using both simulated and real vector data related to the mechanical engineering problem described in Section 4 Note that experiments done for scalar signals are described by Basseville *et al* (1986a) and lead to the conclusion that it is possible to detect and diagnose small changes (1%) in eigenfrequencies provided that the damping coefficients are low, and that the record sizes are high enough (several thousands)

According to Section 4 2, the simulation model which was used is a tied down system of 18 masses of one degree of freedom connected by springs, as shown in Fig 1, with known weights, stiffness and damping coefficients Six-dimensional signals (displacements of the masses 1, 7, 13, 6, 12, 18) have been generated under different hypotheses, including no change, change in mass no 1, change in the stiffness of the connection to the ground, and change in the stiffness of the connection 8–11 These physically different changes are indicated by arrows on Fig 1

The global test t_0 (16) and nine sensitivity tests using (34) have been computed, corresponding to the nine modes of lowest order, considered together with the pulsation ω_j as mentioned at the end of Section 4 These test computations have been done for all the subsets of components corresponding to $r = 2, 3, 4, 6$ and to at least one observation of each of the two opposite "legs" of the structure

For simulated as well as real data, the reference AR model θ_0, to be used for the tests, has been computed by solving (29) where Φ and Δ contain the available eigencharacteristics (a small number

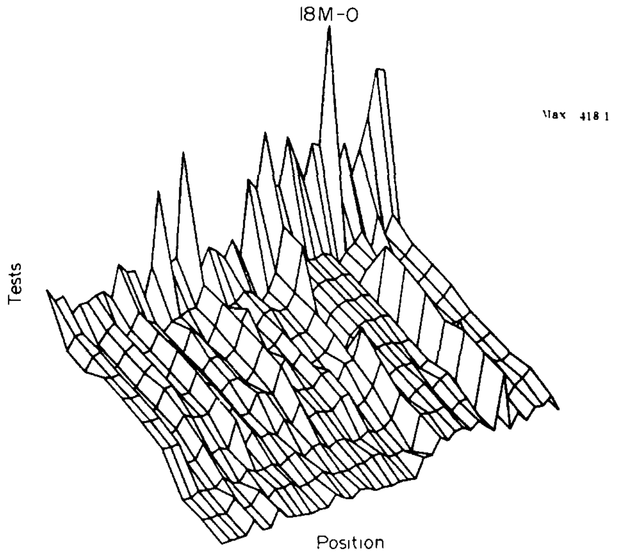

FIG 2 The tests under H_0

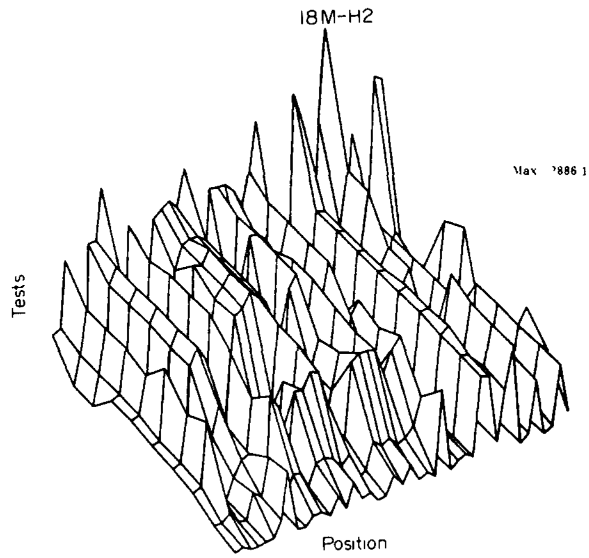

FIG 4 A change in the stiffness of the connection to the ground

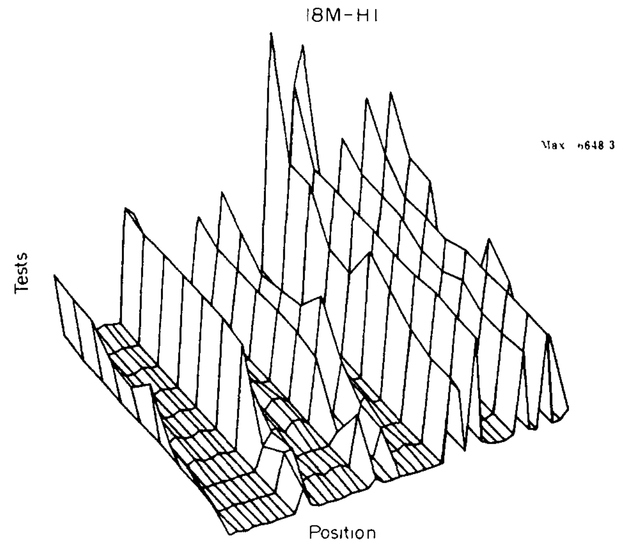

FIG 3 A change in mass 1

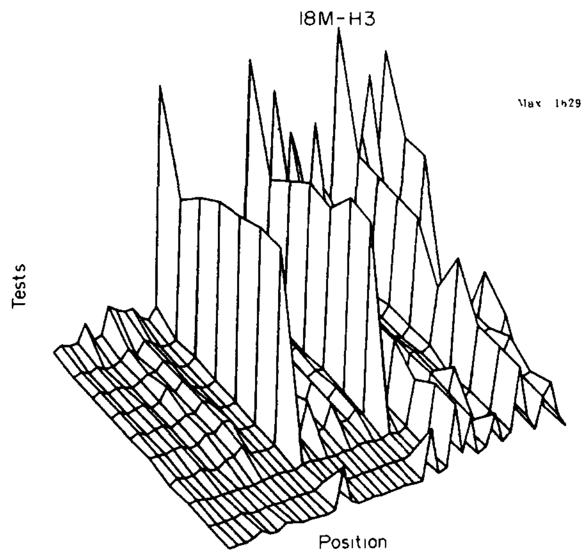

FIG 5 A change in the connection 8–11

for real data) The AR order p is chosen to be the integer part of $2m/r$, and not such that $2m/r \leqslant p < 2m/r + 1$ as done for the identification task (i) of Section 4 The reason for doing so is that it is intuitively more sensible to monitor the structure without introducing other modes than those which are available The chosen number of instruments is $N = p$, for reasons which are discussed in Basseville *et al* (1986a), and therefore t_0 is computed with the aid of (18)

Finally, the Jacobians J_β of Section 4 3 are computed with the aid of (31), where the eigenvalues and eigenvectors which are used are the *available* eigencharacteristics, and not the modes associated with the computed A_i

The results are presented in the three-dimensional Figs 2–5, one horizontal axis corresponding to the different sensor locations and numbers, from two to six from left to right, and the other horizontal axis corresponding to various tests, with the global test in the back and the nine sensitivity tests from the back to the front in the order of increasing frequencies As the pictures were drawn with automatic scaling, the maximum value is indicated Figure 2 shows the behavior of the various tests under the no change hypothesis, Fig 3 corresponds to a change (14%) in mass 1, Fig 4 to a change (12%) in stiffness of ground connection, and Fig 5 to a cancellation of connection 8–11 (which was small) These results show that some sensor locations may be very poor, in the sense that no detection is possible (nearly the same value of the tests under H_0 and H_1), and therefore moving sensors (along legs) may be of interest On the other hand, the visually significantly different profiles obtained in Figs 3–5 lead to the conclusion that these types of tests contain information which allows discrimination among physically different changes An attempt to get direct physical diagnosis is currently under development

6 CONCLUSION

The authors have presented an original approach for solving the two problems of detection and diagnosis of changes in the eigenstructure of nonstationary multivariable systems This approach is based upon the use of instrumental statistics which reduces the detection problem to a problem of change in the mean of a Gaussian process with known covariance matrix The diagnosis problem is solved in the same manner, with the aid of first order approximations relating the changes in the system parameters to be monitored to the changes in the model parameters which are used for monitoring

The application of this methodology to the mechanical engineering problem of vibration monitoring has been described for complex vibrating structures subject to natural nonstationary and uncontrolled excitation Experimental results have been presented, which show the efficiency of the proposed tests, for detecting small changes (typically a few per cent in eigenfrequencies) and for discriminating different physical changes The authors' opinion is that such an approach for diagnosis ("sensitivity" method) is general enough for a possible direct use on physical parameters This approach is currently under study

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APPENDIX A ASSUMPTIONS AND THEOREMS STATEMENT

The precise assumptions and theorems involved in the authors' detection approach are stated here

Let (Y_t) be a nonstationary multivariable process having as state space representation

$$\begin{cases} Y_{t+1} = FX_t + V_{t+1} \\ Y_t = HX_t \end{cases} \tag{A1}$$

where $X_t \in \mathscr{R}^n$, $Y_t \in \mathscr{R}^r$, $\mathrm{cov}(V_{t+1}) = Q_t$, and (V_t) is a white noise with time varying covariance matrix

The assumptions concerning the model (A1) are as follows

(C1) F is full rank and asymptotically stable

(C2) There exists $k > 0$ such that, for any vector u and any integer t,

$$E(u^T V_t)^4 < k \|u\|^4$$

(C3) (H, F) is observable, i e there exists an integer p such that

$$\mathcal{O}_p(H,F) \triangleq \begin{pmatrix} H \\ HF \\ \vdots \\ HF^{p-1} \end{pmatrix}$$

is of rank n

(C4) There exists a matrix G of rank r such that, for any integer t

$$Q_t \geqslant GG^T$$

This last assumption, which is *not* a controllability condition, allows changes in the direction of the excitation

If the nominal model (H_0, F_0) *fulfills the conditions C1–C4, then the two following theorems hold*

(1) *Nonstationary law of large numbers* $\Sigma_N(s)$ given by (10a) is asymptotically uniformly positive definite and bounded, and the estimate $\hat{\Sigma}_N(s)$ given by (10b) is consistent

$$\Sigma_N(s)^{-1} \hat{\Sigma}_N(s) \underset{s\to\infty}{\to} I \quad \text{a s}$$

under both the no change hypothesis H_0 $\theta = \theta_0$, and the small change hypothesis

$$H_1 \quad \theta = \theta_0 + \frac{\delta\theta}{\sqrt{s}}$$

(2) *Nonstationary central limit theorem* Under H_0, $\Sigma_N^{-1/2}(s)\mathscr{U}_N(s) \underset{s\to\infty}{\overset{\mathscr{L}}{\to}} \mathcal{N}(0, I)$ where

$\mathscr{U}_N(s)$ is given by (9)

Under H_1, $\Sigma_N^{-1/2}(s)\,(\mathscr{U}_N(s) - (1/s\,\mathscr{H}_{p\,N}^T(s) \otimes I_r)\delta\Theta) \underset{s\to\infty}{\overset{\mathscr{L}}{\to}} \mathcal{N}(0, I)$

If, further, it is assumed that the following condition is fulfilled

(C5) $$\liminf_{s\to\infty} \sigma_n\left(\frac{1}{s}\mathscr{H}_{p\,N}(s)\right) > \sigma > 0 \quad \text{a s}$$

where $\sigma_n(M)$ is the nth singular value of M, then with (C3) one can ensure the uniform minimality of the state space model (A1) to represent (Y_t)

The assumptions (C1)–(C5) imply the assumptions needed by Benveniste *et al* (1985) for the consistency of the identification, and also imply that the factorization (12) of the Hankel matrix holds

APPENDIX B DIFFERENTIATION OF A MATRIX POLYNOMIAL

Here the details of the computations needed from (30) to (31) are given, related to the Jacobians to be used for diagnosis Rewriting (30)

$$(\mathcal{C}_p(L\phi, e^{\Delta})^T \otimes I_r)\,d\Theta = -\sum_{j=0}^{p} (e^{(p-j)\Delta} \otimes A_j L)\,\mathrm{col}(d\Phi)$$

$$-\sum_{j=0}^{p-1} (I_m \otimes (p-j)A_j L\phi\, e^{(p-j)\Delta})\,\mathrm{col}(d\Delta)$$

$$\triangleq -(J_1 + J_2) \qquad \text{(B1)}$$

First consider the term J_1

$$-\sum_{j=0}^{p} (e^{(p-j)\Delta} \otimes A_j L) = -\sum_{j=0}^{p} \begin{pmatrix} e^{(p-j)\lambda_1} A_j L & & 0 \\ & \ddots & \\ 0 & & e^{(p-j)\lambda_m} A_j L \end{pmatrix} = \begin{pmatrix} A(\mu_1)L & & 0 \\ & \ddots & \\ 0 & & A(\mu_m)L \end{pmatrix}$$

where

$$A(X) = X^p I_r - A_1 X^{p-1} - \cdots - A_{p-1}X - A_p$$

and

$$\mu_j = e^{\lambda_j}$$

Thus

$$J_1 = \begin{pmatrix} A(\mu_1) & & 0 \\ & \ddots & \\ 0 & & A(\mu_m) \end{pmatrix} \begin{pmatrix} L\,d\phi_1 \\ \vdots \\ L\,d\phi_m \end{pmatrix} \qquad \text{(B2)}$$

On the other hand, for computing J_2, the following is needed

$$\mathrm{Col}(d\Delta) = \mathrm{Col}\begin{pmatrix} d\lambda_1 & & 0 \\ & \ddots & \\ 0 & & d\lambda_m \end{pmatrix} = \begin{pmatrix} d\lambda_1 & e_1 \\ \vdots & \vdots \\ d\lambda_m & e_m \end{pmatrix}$$

where e_j is the jth vector of the canonical basis of $\mathscr{R}^m$ Therefore

$$J_2 = -\sum_{j=0}^{p-1} \begin{pmatrix} (p-j)A_j L\phi\, e^{(p-j)\Delta} & & 0 \\ & \ddots & \\ 0 & & (p-j)A_j L\phi\, e^{(p-j)\Delta} \end{pmatrix} \begin{pmatrix} e_1 & & 0 \\ & \ddots & \\ 0 & & e_m \end{pmatrix} \begin{pmatrix} d\lambda_1 \\ \vdots \\ d\lambda_m \end{pmatrix}$$

$$= -\sum_{j=0}^{p-1} \begin{pmatrix} (p-j)A_j L\phi_1\, e^{(p-j)\lambda_1} & & 0 \\ & \ddots & \\ 0 & & (p-j)A_j L\phi_m\, e^{(p-j)\lambda_m} \end{pmatrix} \begin{pmatrix} d\lambda_1 \\ \vdots \\ d\lambda_m \end{pmatrix}$$

$$= \begin{pmatrix} \mu_1 A'(\mu_1)L\phi_1 & & 0 \\ & \ddots & \\ 0 & & \mu_m A'(\mu_m)L\phi_m \end{pmatrix} \begin{pmatrix} d\lambda_1 \\ \vdots \\ d\lambda_m \end{pmatrix} \qquad \text{(B3)}$$

where $A'(X) = pX^{p-1}I_r - (p-1)A_1 X^{p-2} - \cdots - A_{p-1}$

Introducing (B2) and (B3) into (B1) gives

$$(\mathcal{C}_p(L\phi, e^{\Delta})^T \otimes I_r)\,d\Theta = J_\psi\, d\psi + J_\lambda\, d\lambda$$

where $J_\psi, J_\lambda, d\psi, d\lambda$ are defined below formula (31)

Finally, using the observability matrix in the modal basis

$$\mathcal{O}_p^* = \mathcal{C}_p(H,F)\Psi$$

where Ψ is defined in (27) gives (31)