Minimax Equalization for Random Signals

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Abstract—**The design of a fixed filter is considered for equalization of an imprecisely known channel. The channel frequency response is assumed to have amplitude and phase characteristics lying within specified bounds at each frequency, and a minimax filter optimizing worst case mean-squared error (MSE) performance is derived. The general result is illustrated by considering a two-path channel model with an uncertain secondary path delay characteristic.**

I. Introduction

Channel equalization is necessary in many communication systems where the channel characteristics cannot be assumed to be ideal so that linear amplitude and phase distortion occurs. In most cases, the channel characteristic is not simply nonideal, but it is also not precisely known and may be time varying. One approach for equalization under such condiions, which has been widely applied, is to use an adaptive scheme. There do arise situations, however, in which adaptive equalization may not be practical because of cost and complexity and the requirement to adapt rapidly to changing conditions. It may be desirable for such situations to use a fixed equalizer, if one can be designed which gives acceptable performance over the whole range of anticipated channel conditions that may be encountered.

In this paper, we apply the *minimax* formulation to the problem of fixed equalizer design for uncertain channel characteristics. For this, we define a class of possible channel characteristics, and we seek the equalizer which optimizes worst case system performance. We will assume that the signal input to the channel is a stationary random signal with a known power spectral density (PSD), and that the channel output is observed in additive stationary noise with a known PSD. The performance measure will be the mean-squared error (MSE) between the desired signal at the channel input and the output of the equalizer.

We will consider a specific structure for the class of possible channel characteristics, which is quite reasonable and useful in applications. The channel will be modeled as being time invariant so that it can be characterized by a frequency response characteristic $H(\omega)$. The class of allowable characteristics will be defined to contain all $H(\omega)$ with amplitude characteristics $|H(\omega)|$ bounded by known upper and lower bounding functions, and with phase characteristics $\arg\{H(\omega)\}$ whose values are constrained to be, for each frequency ω, in known subsets $\Phi(\omega)$ of $(-\pi, \pi]$. We will obtain explicitly the minimax equalizer for such classes and show that it has an interesting interpretation. At frequencies where the signal-to-noise ratio (SNR) is high relative to a measure of channel certainty, the minimax equalizer essentially inverts the "nominal" channel. At lower SNR frequencies, the minimax equalizer acts as a Wiener filter. We also present a numerical example to illustrate the results.

It is interesting to note that the problem of fixed equalizer design was considered in [1] using a statistical approach for an ensemble of random channels. A more general cost function was used in [1], allowing optimum amplitude scaling in the MSE expression. In addition, the optimum fixed equalizer was sought subject to an output power constraint. In contrast, we use here the minimax approach and a simpler performance index (MSE) without an output power constraint. We are able to find explicit solutions which in their general behavior are related to some of the solutions described in [1]. While it is possible to define performance criteria other than the MSE which may be more appropriate in specific applications, the MSE criterion has the strong appeal of leading to mathematically tractable analysis. It has been widely used to obtain designs for equalizers for data communications [2], and is a resonable criterion to use for analog signals as in the present case.

The minimax approach that we follow in this paper was motivated by recent work on minimax robust signal processing (see, for example, [3] and references therein). Most of this recent work has been concerned with robustness against uncertainties in signal and noise characteristics, whereas here we assume known signal and noise PSD's, but an imprecisely known channel characteristic. We should also note that while our primary motivation in this development is the equalization of imprecisely known communication channels, the results we develop are applicable to the broader class of problems of deconvolution of noisy observations produced by imprecisely known frequency response characteristics. For example, one application is in the restoration of noisy, blurred images. By including signal and noise PSD uncertainties in this problem formation, a more general set of results could be obtained which would be more closely related to the minimax robust signal processing results of the type discussed above.

II. Problem Formulation

Fig. 1 shows the system under consideration. The input to the equalizer is a linearly distorted version of an original finite-power signal $s(t)$, together with random noise $n(t)$. The signal $s(t)$ and the noise $n(t)$ are uncorrelated, zero-mean, stationary random processes with *known* respective PSD's $S(\omega)$ and $N(\omega)$. Suppose an equalizer with frequency response $G(\omega)$ is used when the time-invariant channel has frequency response $H(\omega)$. Then the MSE $e(H, G)$ between the system output $\hat{s}(t)$ and the original signal $s(t)$ can easily be shown to be

$$e(H, G) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \{|1 - H(\omega)G(\omega)|^2 S(\omega) + |G(\omega)|^2 N(\omega)\} \, d\omega. \tag{1}$$

Notice that at ω values where both $S(\omega)$ and $N(\omega)$ are zero, there is no contribution to the MSE. We will assume that $S(\omega)$ and $N(\omega)$ are not both zero at any ω.

We assume that the channel amplitude characteristic is a measurable function and is bounded by two known measur-

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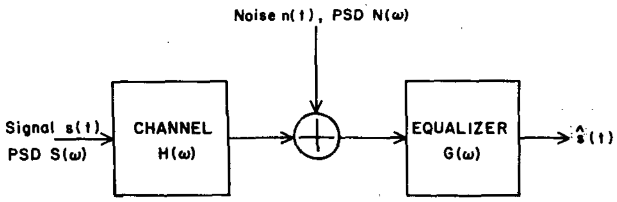

Fig. 1. Equalization of noisy linear channel.

able functions $A_L(\omega)$ and $A_U(\omega)$, so that

$$A_L(\omega) \leqslant |H(\omega)| \leqslant A_U(\omega). \tag{2}$$

The lower bounding function is nonnegative, and the upper bounding function $A_U(\omega)$ will be taken to be positive without loss of generality, and to have a finite upper bound $A_U(\omega) \leqslant C < \infty$. This is clearly reasonable; in fact, we have $C \leqslant 1$ for passive channels. Let the phase characteristic arg $\{H(\omega)\}$ of the channel be a measurable function denoted by $\phi(\omega)$. We also assume that for each ω, there is a known closed *subset* $\Phi(\omega)$ of $(-\pi, \pi]$ which contains the value of $\phi(\omega)$. Later we will make explicit one more condition on the allowable $\phi(\omega)$ to ensure that the mathematical derivations in the next section are valid. Thus, $A_L(\omega)$, $A_U(\omega)$, and $\Phi(\omega)$ together define a class $\mathcal{H}$ of allowable channel characteristics.

Let $\mathcal{G}$ denote the class of all possible equalizer frequency responses. Then our objective is to find the minimax filter frequency response $G_M(\omega)$ which satisfies

$$\min_{G \in \mathcal{G}} \max_{H \in \mathcal{H}} e(H, G) = \max_{H \in \mathcal{H}} e(H, G_M). \tag{3}$$

Thus, $G_M(\omega)$ will be the frequency response of a minimax equalization filter which optimizes worst-case estimation performance for the class $\mathcal{H}$ of allowable channel characteristics.

Our approach will be to consider the *integrand* on the right-hand side of (1), and to obtain a minimax solution by considering this integrand *pointwise* for each ω. We will find that the minimax filter frequency response obtained by this pointwise optimization and the corresponding worst case channel frequency response are well-behaved functions of ω (e.g., continuous) as long as $S(\omega)$, $N(\omega)$, $A_L(\omega)$, $A_U(\omega)$, and the functions characterizing the subsets $\Phi(\omega)$ are well-behaved functions (e.g., piecewise continuous, bounded). In addition, the MSE of (1) will always be well defined and bounded for all channel characteristics in $\mathcal{H}$ for the minimax equalizer frequency response we will derive by our approach. It is true, of course, that because of the pointwise constraints on members of $\mathcal{H}$, no smoothness restrictions are imposed on them. This means that the class of allowable channel characteristics treated in this approach is generally much larger than would be obtained under, say, a continuity requirement. Such a restriction would make the pointwise considerations invalid, and make it very difficult to obtain an explicit solution. We will see, however, as noted above, that the minimax equalizer derived by our approach will, in situations of practical interest, be quite well behaved. This will be seen, in particular, for the example in Section IV. Thus, in practical situations, it will generally be possible to obtain good approximations to the characteristics of the ideal minimax equalizer we will derive. In addition, the fact that in such situations the minimax equalizer frequency response generally turns out to be well behaved means that it is optimum for some quite reasonable and nonpathological member of the class of allowable channel frequency responses, so that the pathological members of $\mathcal{H}$ do not influence the solution for the ideal minimax equalizer.

III. Explicit Solution for Minimax Filter

We first obtain the channel characteristic $H_G(\omega)$ which maximizes the MSE $e(H, G)$ for a given equalizer characteristic $G(\omega)$. To do this, we maximize *pointwise* the term $|1 - H(\omega)G(\omega)|^2$ in the integrand in (1). Now this can be written as

$$|1 - H(\omega)G(\omega)|^2 = 1 - 2|H(\omega)||G(\omega)| \cdot \cos[\phi(\omega) + \theta(\omega)] + |H(\omega)|^2|G(\omega)|^2 \tag{4}$$

where $\theta(\omega)$ is the phase characteristic arg $\{G(\omega)\}$ of the equalizer. For given $|G(\omega)|$ and $\theta(\omega)$, we see from (4) that $|1 - H(\omega)G(\omega)|^2$ is maximized with $\phi(\omega)$ chosen as

$$\phi_G(\omega) = \arg \min_{\phi(\omega) \in \Phi(\omega)} |[\phi(\omega) + \theta(\omega)] \bmod (2\pi) - \pi|. \tag{5}$$

This function is always well defined because the function $|x \bmod (2\pi) - \pi|$ is lower semicontinuous and $\Phi(\omega)$ is compact. This means that the characteristic $H_G(\omega)$ maximizing the integrand of $e(H, G)$ for a given $G(\omega) = |G(\omega)| \exp[j\theta(\omega)]$ has a phase function $\phi(\omega) = \phi_G(\omega)$ which minimizes $|[\phi(\omega) + \theta(\omega)] \bmod (2\pi) - \pi|$ for each ω over the set $\Phi(\omega)$ of allowable phase values. With this choice for the phase function of the channel, further maximization of the right-hand side of (4) with respect to the channel amplitude characteristic for fixed $|G(\omega)|$ yields the result

$$|H_G(\omega)| = \begin{cases} A_L(\omega), & \cos[\phi_G(\omega) + \theta(\omega)] \\ & \geqslant |G(\omega)|[A_L(\omega) + A_U(\omega)]/2 \\ A_U(\omega), & \text{otherwise.} \end{cases} \tag{6}$$

This follows from the fact that the right-hand side of (4) is a quadratic expression in $|H(\omega)|$ and has a *minimum* at $|G(\omega)| \cdot |H(\omega)| = \cos[\phi(\omega) + \theta(\omega)]$.

We now have the phase and amplitude characteristics $\phi_G(\omega)$ and $|H_G(\omega)|$ of a function $H_G(\omega)$ which maximizes the integrand of $e(H, G)$ pointwise for given equalizer characteristic $G(\omega)$. The resulting integrand is a function of the equalizer amplitude characteristic $|G(\omega)|$ and the equalizer phase characteristic $\theta(\omega)$ and is given by

$$I = \{1 - 2|H_G(\omega)||G(\omega)| \cos[\phi_G(\omega) + \theta(\omega)] + |H_G(\omega)|^2|G(\omega)|^2\}S(\omega) + |G(\omega)|^2 N(\omega). \tag{7}$$

To minimize I over the class $\mathcal{G}$ of equalizer frequency responses, let us consider the phase characteristic $\theta(\omega)$ first. Note that in the integrand I, the phase $\phi_G(\omega)$ *minimizes* $\cos[\phi(\omega) + \theta(\omega)]$ for given $\theta(\omega)$ over the set $\Phi(\omega)$. Therefore, to minimize I with respect to the phase characteristic $\theta(\omega)$, we need to choose $\theta(\omega)$ to *maximize* the *minimum* value $\cos[\phi_G(\omega) + \theta(\omega)]$. Let us interpret $\Phi(\omega)$ as some set of points on the unit circle in the complex plane, with $2\alpha(\omega)$ the angle subtended at the origin by the largest arc(s) on the unit circle *outside* $\Phi(\omega)$. Let $\beta(\omega)$ be the angular location of the middle of such an arc (see Fig. 2). We will assume that $\alpha(\omega)$ and $\beta(\omega)$ are measurable functions.[1] Then it follows that an equalizer phase $\theta_M(\omega)$ minimizing I for given $|G(\omega)|$ is

$$\theta_M(\omega) = \pi - \beta(\omega). \tag{8}$$

[1] This can be obtained, for example, when $\Phi(\omega)$ is a finite union of closed intervals whose endpoints are measurable functions of ω.

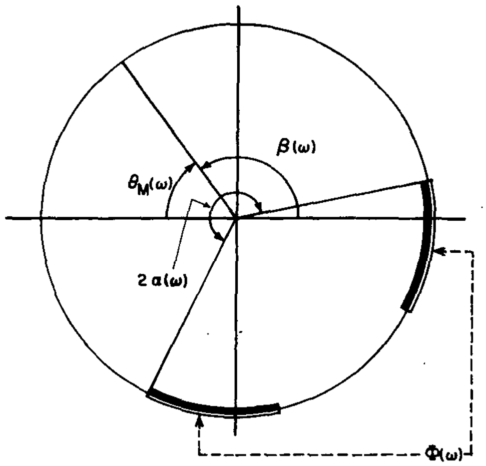

Fig. 2. Illustration of the uncertainty sets $\Phi(\omega)$ and the angles $\alpha(\omega)$, $\beta(\omega)$, and $\theta_M(\omega)$.

The angle $\beta(\omega)$ may not be unique. The phase $\theta_M(\omega)$ can be interpreted as that phase which, when added to $\beta(\omega)$, transforms it into the point π on the unit circle. With this phase angle, the integrand I of (7) becomes

$$\begin{aligned} I &= \{1 + 2|H_G(\omega)||G(\omega)| \cos [\alpha(\omega)] \\ &\quad + |H_G(\omega)|^2 |G(\omega)|^2\} S(\omega) + |G(\omega)|^2 N(\omega) \\ &= S(\omega) + \{2S(\omega)|H_G(\omega)| \cos [\alpha(\omega)]\} |G(\omega)| \\ &\quad + \{|H_G(\omega)|^2 S(\omega) + N(\omega)\} |G(\omega)|^2. \end{aligned} \tag{9}$$

We finally consider minimization of the integrand I given by (9) over the amplitude characteristic $|G(\omega)|$ of the equalizer. Note first that $\alpha(\omega)$ must be between 0 and π rad, and that if $\cos [\alpha(\omega)]$ is nonnegative (i.e., $0 \leqslant \alpha(\omega) \leqslant \pi/2$), then $|G(\omega)| = 0$ minimizes I. Consider, therefore, the case $\cos [\alpha(\omega)] < 0$.

Now in the integrand I, the quantity $|H_G(\omega)|$ is given by (6) so that it is either $A_L(\omega)$ or $A_U(\omega)$. When $|G(\omega)| = 0$, the integrand I is always $S(\omega)$, and when $|G(\omega)| = -2 \cos [\alpha(\omega)]/[A_L(\omega) + A_U(\omega)]$, we have its common value I_c given by

$$\begin{aligned} I_c &= S(\omega) \left\{1 - \frac{4 \cos^2 [\alpha(\omega)] A_L(\omega) A_U(\omega)}{[A_L(\omega) + A_U(\omega)]^2}\right\} \\ &\quad + N(\omega) \frac{4 \cos^2 [\alpha(\omega)]}{[A_L(\omega) + A_U(\omega)]^2}, \end{aligned} \tag{10}$$

which is nonnegative. Let us denote by $I_U[|G(\omega)|]$ the function of $|G(\omega)|$ we get for I with $A_U(\omega)$ in place of $|H_G(\omega)|$ in (9), and let $I_L[|G(\omega)|]$ be the function obtained with $A_L(\omega)$ replacing $|H_G(\omega)|$ in (9). Both I_U and I_L are quadratic expressions in $|G(\omega)|$, with I_U having a minimum at a value of its argument $|G(\omega)|$ which is less than or equal to $-2 \cos [\alpha(\omega)]/[A_L(\omega) + A_U(\omega)]$. This follows from the fact that the value of $|G(\omega)|$ minimizing $I_U[|G(\omega)|]$ is

$$\begin{aligned} |G(\omega)|_{U,m} &= \frac{-S(\omega) A_U(\omega) \cos [\alpha(\omega)]}{A_U^2(\omega) S(\omega) + N(\omega)} \leqslant \frac{-\cos [\alpha(\omega)]}{A_U(\omega)} \\ &\leqslant \frac{-2 \cos [\alpha(\omega)]}{A_L(\omega) + A_U(\omega)}. \end{aligned} \tag{11}$$

The minimum value of $I_L[|G(\omega)|]$ occurs at the argument value

$$|G(\omega)|_{L,m} = \frac{-S(\omega) A_L(\omega) \cos [\alpha(\omega)]}{A_L^2(\omega) S(\omega) + N(\omega)}, \tag{12}$$

provided $A_L(\omega)S(\omega)$ and $N(\omega)$ are not both zero.[2] When $A_L(\omega)S(\omega)$ and $N(\omega)$ are both zero, we note that $I_L[|G(\omega)|] = S(\omega)$ for any value of $|G(\omega)|$.

From the above considerations, we find that $I[|G(\omega)|]$ of (9) is described by either Fig. 3(a) or (b) since it is given by

$$I[|G(\omega)|] = \begin{cases} I_L[|G(\omega)|], & |G(\omega)| \leqslant \dfrac{-2 \cos [\alpha(\omega)]}{A_L(\omega) + A_U(\omega)} \\ I_U[|G(\omega)|], & |G(\omega)| \geqslant \dfrac{-2 \cos [\alpha(\omega)]}{A_L(\omega) + A_U(\omega)}. \end{cases} \tag{13}$$

Thus, we are finally led to the amplitude characteristic $|G_M(\omega)|$ of the minimax equalizer $G_M(\omega)$ by choosing the amplitude minimizing $I[|G(\omega)|]$; it is given by

$$|G_M(\omega)| = \begin{cases} 0, & \cos [\alpha(\omega)] \geqslant 0 \\ \dfrac{-2 \cos [\alpha(\omega)]}{A_L(\omega) + A_U(\omega)}, & \dfrac{A_L^2(\omega) S(\omega)}{N(\omega)} \geqslant \dfrac{2A_L(\omega)}{[A_U(\omega) - A_L(\omega)]} \\ & \text{and } \cos [\alpha(\omega)] < 0 \\ \dfrac{-S(\omega) A_L(\omega) \cos [\alpha(\omega)]}{A_L^2(\omega) S(\omega) + N(\omega)}, & \text{otherwise.} \end{cases} \tag{14}$$

This expression is valid when $A_L(\omega)S(\omega)$ and $N(\omega)$ are not both zero. (When they are both zero, any $|G_M(\omega)| \leqslant -2 \cdot \cos [\alpha(\omega)]/A_U(\omega)$ gives a minimax solution.) Further, we have assumed $A_U(\omega) > 0$; otherwise, we have $|G_M(\omega)| = 0$. The angle $\alpha(\omega)$ is always uniquely defined in terms of the set $\Phi(\omega)$ from Fig. 2. The angle $\beta(\omega)$ which gives the phase $\theta_M(\omega)$ of the minimax filter from (8) may not be unique. We can show the validity of our results for the minimax equalizer by reversing the order in which the minimization of the integrand of $e(H_G, G)$ was performed, that is, by considering $|G(\omega)|$ first and then $\theta(\omega)$, which gives the same minimax equalizer. It is easy to show that under the assumptions we have made in deriving the above results, the MSE $e(H, G_M)$ of (1) is always well defined and bounded for all $H \in \mathcal{H}$; the upper bound is, of course, obtained with $H(\omega)$ defined by (6) and (5) for $G(\omega) = G_M(\omega)$.

The amplitude characteristic $|G_M(\omega)|$ of the minimax equalizer is zero when the phase uncertainty is large. Note that $[2\pi - 2\alpha(\omega)]/2\pi$ or $1 - \alpha(\omega)/\pi$ is a measure of this *uncertainty*, and $1 - \alpha(\omega)/\pi > 1/2$ leads to the use of a minimax equalizer with zero output, the mean value of the signal. For smaller phase uncertainty, the equalizer acts essentially as either the inverse of a "nominal" channel with amplitude characteristic $[A_L(\omega) + A_U(\omega)]/2$ or as a Wiener filter. Note that we may define a measure of the *uncertainty* $\delta(\omega)$ about

[2] This condition is satisfied if an arbitrarily small white noise component is present.

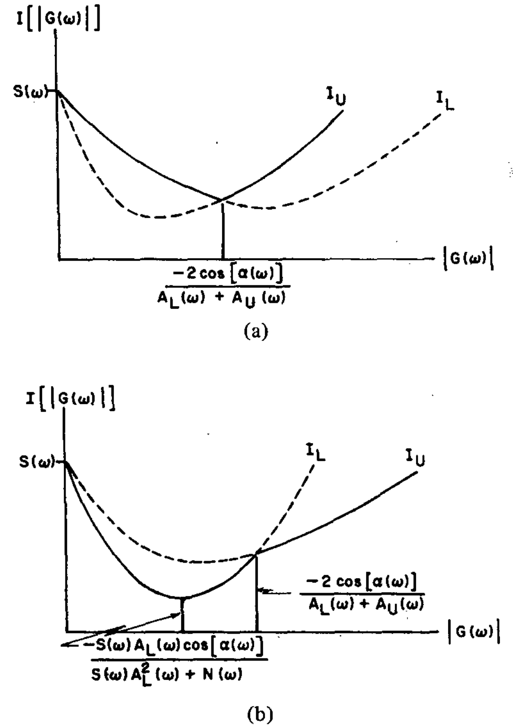

Fig. 3. Minimization of $I(|G(\omega)|)$ for solution of minimax filter amplitude [see (13)].

the channel amplitude characteristic at frequency ω by

$$\delta(\omega) = \frac{A_U(\omega) - A_L(\omega)}{A_U(\omega) + A_L(\omega)}. \tag{15}$$

With this definition, we find that

$$\frac{1 - \delta(\omega)}{\delta(\omega)} = \frac{2A_L(\omega)}{A_U(\omega) - A_L(\omega)} \tag{16}$$

where the quantity on the right-hand side of (16) above appears in the expression (14) for the amplitude characteristic of the minimax equalizer. We may thus interpret it as a degree of *certainty* one has about the channel amplitude characteristic (measured on a scale from zero to infinity).

Thus, the minimax equalizer acts as the inverse of a "nominal" channel, with another attenuation factor $-\cos[\alpha(\omega)]$ due to phase uncertainty when the minimum SNR $A_L{}^2(\omega)S(\omega)/N(\omega)$ at the equalizer input is larger than or equal to a measure of the degree of certainty about the channel amplitude characteristic. Otherwise, when the contamination of the signal due to additive noise is the dominant degradation relative to uncertainty about the channel amplitude characteristic, the equalizer acts as a Wiener filter for the lowest gain channel. Once again, an additional attenuation $-\cos[\alpha(\omega)]$ arises because of the phase uncertainty. In both cases, the phase of the minimax equalizer is the conjugate phase of a "nominal" channel, as seen from Fig. 2 and the result (8).

If there is no channel uncertainty, so that $A_L(\omega) = A_U(\omega) = |H(\omega)|$ and the set $\Phi(\omega)$ is composed of a single point $\phi(\omega)$ for each ω, (8) and (14) yield for the optimum equalizer $G_{\rm opt}(\omega)$ the classical Wiener filter characteristic

$$G_{\rm opt}(\omega) = \frac{S(\omega)H^*(\omega)}{|H(\omega)|^2 S(\omega) + N(\omega)}. \tag{17}$$

IV. Multipath Channel Example

To illustrate the use of our result, let us consider equalization of a two-path channel. We will let the primary communication channel have an ideal (unity) frequency response, and assume that the secondary channel has an imprecisely known delay characteristic. The two-path channel will be modeled as having a frequency response

$$H(\omega) = 1 + a_0 e^{-j\omega\tau(\omega)} \tag{18}$$

where a_0 and $\tau(\omega)$ are the secondary channel constant gain and frequency-dependent delay characteristic, respectively. We assume that the gain a_0 is known and satisfies $0 < a_0 < 1$. The delay characteristic will, however, be assumed to satisfy the constraint

$$0 \leqslant \tau(\omega) \leqslant \tau_0 \tag{19}$$

for all values of ω where τ_0 is a known finite upper bound on $\tau(\omega)$. It is possible to consider a more general situation where the gain of the secondary channel is constrained only to have known lower and upper bounds, and to make the lower bound in (19) more generally some value other than zero. The simple case we consider here allows us to illustrate more clearly the application and usefulness of the general result.

We will assume that the signal has an ideal low-pass PSD,

$$S(\omega) = \begin{cases} S_0, & |\omega| \leqslant \omega_0 \\ 0, & |\omega| > \omega_0 \end{cases} \tag{20}$$

and that the noise is white with PSD N_0. For convenience, we define the noise-to-signal ratio $K \triangleq N_0/S_0$.

Now from our channel model, we can conclude that $|H(\omega)| \leqslant 1 + a_0$, so that we have $A_U(\omega) = 1 + a_0$ here. From Fig. 4, it also follows quite easily that the lower bound $A_L(\omega)$ for $|H(\omega)|$ is

$$A_L(\omega) = \begin{cases} [1 + 2a_0 \cos(x) + a_0{}^2]^{1/2}, & |x| \leqslant \pi \\ 1 - a_0, & \text{otherwise} \end{cases} \tag{21}$$

where $x \triangleq \omega\tau_0$. Without using these amplitude constraints, that is, by considering only extreme values for the channel phase separately, we find that the sets $\Phi(\omega)$ of allowable phase angles are intervals $[\phi_L(\omega), \phi_U(\omega)]$ with boundaries

$$\phi_L(\omega) = \begin{cases} -\sin^{-1}\left\{\dfrac{a_0 \sin(x)}{[1 + 2a_0 \cos(x) + a_0{}^2]^{1/2}}\right\}, & 0 \leqslant x \leqslant \dfrac{\pi}{2} + \sin^{-1}(a_0) \\ -\sin^{-1}(a_0), & \dfrac{\pi}{2} + \sin^{-1}(a_0) < x \end{cases} \tag{22}$$

and

$$\phi_U(\omega) = \begin{cases} 0, & 0 \leqslant x \leqslant \pi \\ \sin^{-1}\left\{\dfrac{a_0 \sin(x)}{[1 + 2a_0 \cos(x) + a_0{}^2]^{1/2}}\right\}, & \pi < x \leqslant \dfrac{3\pi}{2} - \sin^{-1}(a_0) \\ \sin^{-1}(a_0), & \dfrac{3\pi}{2} - \sin^{-1}(a_0) < x. \end{cases} \tag{23}$$

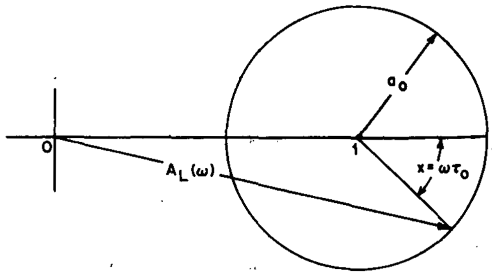

Fig. 4. Geometry for computing amplitude and phase bounds for multipath channel example.

We have, of course, $\phi_L(-\omega) = -\phi_U(\omega)$. The phase characteristic $\theta_M(\omega)$ of the minimax equalizer is the odd-symmetric function $\theta_M(\omega) = -\frac{1}{2}[\phi_L(\omega) + \phi_U(\omega)]$.

The condition which has to be checked in order to obtain $|G_M(\omega)|$ explicitly from (14) is

$$\frac{A_L{}^2(\omega)S(\omega)}{N(\omega)} \geqslant \frac{2A_L(\omega)}{A_U(\omega) - A_L(\omega)}. \tag{24}$$

This becomes, for our particular example,

$$A_L{}^2(\omega) - (1 + a_0)A_L(\omega) + 2K \leqslant 0. \tag{25}$$

Let $A_1 = \{(1 + a_0) + [(1 + a_0)^2 - 8K]^{1/2}\}/2$ and $A_2 = \{(1 + a_0) - [(1 + a_0)^2 - 8K]^{1/2}\}/2$ be the two roots of the quadratic equation obtained with the equality in (25). Then (25) is equivalent to

$$A_2 \leqslant A_L(\omega) \leqslant A_1. \tag{26}$$

If we have $K > (1 + a_0)^2/8$, then $|G_M(\omega)|$ is always defined by the third line of (14) because in this case, (25) can never be satisfied. Since K is the noise-to-signal ratio, this is in agreement with our interpretation that for relatively low SNR, the minimax equalizer acts essentially as a Wiener filter. With $K \leqslant (1 + a_0)^2/8$, real roots $A_1 \leqslant A_2$ exist. Noting that $A_L(\omega)$ of (21) is a monotone nonincreasing function of ω for $\omega \geqslant 0$ with value $(1 + a_0)$ at $\omega = 0$ and value $(1 - a_0)$ at $\omega\tau_0 = \pi$, for $K \leqslant (1 + a_0)^2/8$ we have the following.

1) If $A_2 \leqslant (1 - a_0) \leqslant A_1$, which is true if $K \leqslant a_0(1 - a_0)$, then $|G_M(\omega)|$ is initially defined by the second line of (14) and then, beyond a certain value of ω, is defined by the third line of (14).

2) If $A_1 < (1 - a_0)$, which is true if $a_0 < 1/3$ and $K > a_0(1 - a_0)$, then $|G_M(\omega)|$ is always defined by the third line of (14).

3) If $(1 - a_0) < A_2$, which is true if $a_0 > 1/3$ and $K > a_0(1 - a_0)$, then $|G_M(\omega)|$ is initially defined by the third line in (14), then for intermediate values of ω by the second line of (14), and finally beyond a certain value of ω again by the third line of (14).

If we use our information on the channel characteristics to define an "average" or "nominal" channel

$$H_N(\omega) = 1 + a_0 e^{-j\omega\tau_0/2}, \tag{27}$$

we can define a "nominal" equalizer filter characteristic from (17) as

$$G_N(\omega) = \begin{cases} \dfrac{H_N{}^*(\omega)}{|H_N(\omega)|^2 + K}, & 0 \leqslant |x| \leqslant x_0 \\ 0, & |x| > x_0 \end{cases} \tag{28}$$

where $x_0 \triangleq \omega_0\tau_0$. This would be the naive solution to the fixed equalizer design problem.

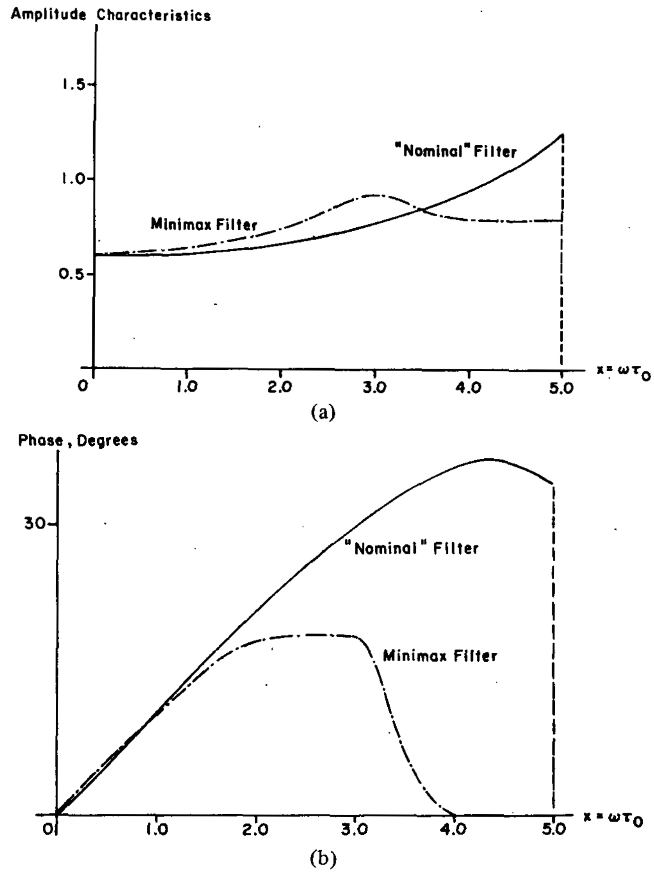

Fig. 5. (a) Amplitude characteristics of nominal and minimax filters ($a_0 = 0.6$, $x_0 = \omega_0\tau_0 = 5$, $N_0/S_0 = 0.1$). (b) Phase characteristics of nominal and minimax filters ($a_0 = 0.6$, $x_0 = \omega_0\tau_0 = 5$, $N_0/S_0 = 0.1$).

Fig. 5(a) and (b) shows the amplitude and phase characteristics of the minimax and "nominal" filters obtained for the parameter values $a_0 = 0.6$, $K = 0.1$, and $x_0 \triangleq \omega_0\tau_0 = 5.0$. For these parameter values, the minimax filter has an amplitude characteristic of the form described by case 1) above.

Table I shows the MSE performance of the minimax and nominal filters computed for three different values of the upper bound τ_0 on the secondary channel delay characteristic $\tau(\omega)$. For each of these three cases, three different values of the secondary channel gain a_0 were used. The noise-to-signal ratio K was taken to be 0.1 or -10 dB. For each combination of the parameters a_0 and τ_0 the MSE's were obtained for three different actual secondary channel delay characteristics of $\tau(\omega) = 0$, $\tau(\omega) = \tau_0/2$, and $\tau(\omega) = \tau_0$. The normalized MSE's shown in the table are the MSE's relative to the signal variance.

Note that the minimax filter design is based on the knowledge of a_0 and the upper bound τ_0, while the "nominal" filter is a Wiener filter for the secondary channel gain a_0 and "nominal" delay characteristic $\tau(\omega) = \tau_0/2$.

Some interesting characteristics are evident from Table I. We find that the "nominal" filter performance can degrade considerably for variations in the actual delay characteristic from its assumed value of $\tau_0/2$. This is particularly evident for the largest value 6 of $\omega_0\tau_0$ (more uncertainty) and for the largest value 0.8 of a_0 (more contribution from the uncertain secondary channel). We also see, and more so for such situations, that the minimax filter normalized MSE performance is relatively quite acceptable under "nominal" conditions (for which the "nominal" filter is optimum), and is significantly better for the nonnominal situations $\tau(\omega) = 0$ and $\tau(\omega) = \tau_0$. The minimax filter has a normalized MSE which does not fluctuate much under different delay conditions, so that its performance is more predictable under channel uncertainty.

TABLE I
NORMALIZED MSE's OF NOMINAL AND MINIMAX FILTERS FOR VARIOUS SECONDARY CHANNEL CONDITIONS; NOISE-TO-SIGNAL RATIO = 0.1

Secondary Channel Gain a_0	Secondary Channel Delay Characteristic $\tau(\omega)$	$\tau_0 = 4.0/\omega_0$		$\tau_0 = 5.0/\omega_0$		$\tau_0 = 6.0/\omega_0$	
		MSE of Nominal Filter	MSE of Minimax Filter	MSE of Nominal Filter	MSE of Minimax Filter	MSE of Nominal Filter	MSE of Minimax Filter
0.4	0	0.17	0.15	0.26	0.16	0.35	0.16
	$\tau_0/2$	0.07	0.10	0.08	0.13	0.10	0.16
	τ_0	0.23	0.17	0.33	0.18	0.42	0.18
0.6	0	0.29	0.20	0.52	0.19	0.81	0.18
	$\tau_0/2$	0.05	0.12	0.07	0.19	0.11	0.24
	τ_0	0.35	0.26	0.62	0.28	0.94	0.27
0.8	0	0.42	0.23	0.85	0.19	1.51	0.17
	$\tau_0/2$	0.05	0.14	0.06	0.24	0.12	0.33
	τ_0	0.48	0.36	0.97	0.39	1.71	0.35

One more observation should be made about the example in this section. We started off with bounds given for the phase characteristic of the *secondary* path, and used these to derive bounds for the amplitude and phase characteristics of the *two-path* channel. These amplitude and phase bounds were obtained independently, so that the result gives a class of channel characteristics which is larger than that specified by the original constraints. Thus, the minimax filter optimizes worst case performance over a class larger than that containing the channel characteristics used in generating the results in Table I. In spite of our enlargement of the original class of channel characteristics to fit the channel uncertainty model for which we obtained the general result in Section III, we find that the minimax filter design is quite useful in maintaining an acceptable level of performance. It should be noted that the "nominal" channel for this example was defined from the original uncertainty class (specified by bounds for the secondary path parameters) and not from the derived uncertainty class for the overall channel.

V. CONCLUSION

In this paper, we applied the minimax criterion to obtain an explicit design for a fixed filter for equalization of a channel whose frequency response characteristics are not precisely known. We used a very reasonable model for the class of allowable channel characteristics, and obtained a result which has an intuitive and interesting interpretation. The minimax equalizer suppressed its input at frequencies for which channel phase uncertainty, as measured by a well-defined quantity, exceeded some threshold. At other frequencies, the minimax filter essentially acted as either: 1) the inverse of a nominal channel for minimum SNR larger than a well-defined measure of *certainty* about the channel or 2) a Wiener filter at frequencies of lower SNR.

The numerical results given in Section IV suggest that the minimax filter is able to maintain a steady performance over entire classes of channel characteristics, whereas the filter optimum for a "nominal" channel only can undergo a large variation in its performance over the same classes. In any given application, the usefulness of the minimax filter will depend on the acceptibility of the performance degradation arising from use of the minimax filter when the "nominal" conditions hold and the seriousness of the performance degradation of the "nominal" filter under mismatch.

A potential exists for extension of these results to the situation where the signal or noise PSD's are not precisely known. Another interesting extension could consider a performance measure which allows optimum amplitude scaling in the MSE expression, as used in [1].

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