New Efficient LS and SVD Based Techniques for High-Resolution Frequency Estimation

George V. Moustakides, *Member, IEEE,* and Kostas Berberidis, *Member, IEEE*

***Abstract*—New least squares and singular value decomposition based methods for the estimation of the frequencies of complex sinusoids in white noise are presented. The methods are based on a new symmetric prediction problem that has some very useful properties leading to algorithms that have considerably reduced complexity. The new symmetric predictor is superior in performance as compared to the well known symmetric Smoother and has a performance comparable to other well known methods. Finally a new LS based method, which combines the new prediction technique with the FBLP method is proposed. This method performs slightly better than the FBLP offering at the same time a significant computational saving. As a by-product in the derivation of the new methods is the establishment of some useful properties concerning the eigenstructure of Hermitian and Persymmetric matrices.**

I. INTRODUCTION

THE problem of estimating the unknown parameters of sinusoidal components embedded in white noise is encountered in many application areas as speech processing, communications, transient analysis, etc. The most important parameters are the frequencies of the sinusoidal components, which once estimated can in turn be used for the computation of the remaining unknown parameters (i.e., amplitudes and phases). Traditional techniques namely the periodogram as well as other Fourier transform based methods fail to resolve closely spaced frequencies. This fact has led to the introduction of a variety of more sophisticated methods which, by fitting linear prediction models to processes consisting of complex sinusoids in noise, result in high resolution spectral line estimation beyond the limits of traditional techniques.

The associated linear predictor may be computed either by using least squares (LS) spectral analysis techniques [1], [3]; or by using singular value decomposition (SVD) based techniques that reduce the effect of noise in the associated autocorrelation matrix [3], [6], [7]. At medium to high SNR both techniques have comparable performance but at medium to low SNR the SVD based methods seem to outperform significantly their LS counterparts. However, the eigenanalysis based methods, due to the required SVD, have a considerably increased computational complexity as compared with the LS ones. This complexity may be prohibitive in some applications especially when real time operation is required. Thus, at medium to high SNR the LS methods are often preferable.

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G. V. Moustakides is with the Department of Computer Engineering and Informatics, University of Patras, Patras, Greece.

K. Berberidis is with the Computer Technology Institute (CTI), Patras, Greece.

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Among the existing LS spectral analysis methods the forward backward linear prediction (FBLP) method (see pp. 222–224 of [1] and pp. 391–394 of [2]) seems superior in the general case.

It is well known that in the case of undamped exponentials in noise the problem leads naturally to the use of conjugate symmetric predictors. Until recently the only known symmetric predictor applicable to this problem was the symmetric Smoother (see pp. 327–331 of [1] and [4], [10]). In [5], a new symmetric predictor, the so-called bidirectional, was shown to be suitable for spectral line estimation. This predictor was defined only for the real data case and there was no study of its performance.

In this paper, first, the LS complex bidirectional predictor is defined. The extension to the complex case is not straightforward and requires the solution of a double minimization problem. Asymptotic results as well as extensive experiments show the superiority of the bidirectional predictor with respect to smoother. This performance is accompanied with a significant computational saving. It is also shown that a proper combination of the complex bidirectional predictor with the FBLP exhibits a superior performance with respect to the FBLP method offering at the same time a significant reduction in computational complexity.

Second, an SVD-based technique for the symmetric prediction problem is developed. Two methods—one for the bidirectional and one for the Smoother are presented. Due to the rich structure of the bidirectional prediction problem the corresponding method performs almost identically as the Minimum Norm method [7] but requiring much less computational burden.

II. THE COMPLEX BIDIRECTIONAL PREDICTOR

Let us assume that we are given a complex data sequence $\{x(n)\}, n = 1, \cdots, N$ consisting of p undamped exponentials in complex white noise, i.e.,

$$x(n) = \sum_{k=1}^{p} A_k e^{j(2\pi f_k n + \phi_k)} + w(n) = \sum_{k=1}^{p} h_k z_k^n + w(n) \quad (1)$$

where $z_k = e^{j2\pi f_k}$ and $h_k = A_k e^{j\phi_k}$. Let us call $P_p(z)$ the polynomial of order p that has as roots the z_k, $k = 1, \cdots, p$, that is

$$P_p(z) = \prod_{k=1}^{p} (1 - z^{-1} z_k) = \sum_{k=0}^{p} a_k z^{-k}. \quad (2)$$

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Since all the roots of $P_p(z)$ lie on the unit circle the coefficients of the polynomial have some very characteristic properties (see pp. 312–315 of [1]). Notice that the coefficient a_p has unit modulus (being the product of all z_k). Thus if $a_p = e^{j2\theta^0}$ and we define $G_p^0(z) = e^{-j\theta^0} P_p(z)$ then it can be shown that the polynomial $G_p^0(z)$ has conjugate symmetric coefficients, namely, if $\boldsymbol{g}_{p+1}^0$ is the vector of its coefficients then

$$\boldsymbol{g}_{p+1}^0 = [e^{-j\theta^0} d_1^0 d_2^0 \cdots d_2^{0*} d_1^{0*} e^{j\theta^0}]^t \tag{3}$$

where $*$ denotes complex conjugate. If we apply the filter $G_p^0(z)$ to data generated as in (1) but without noise, then we conclude that for $n > p$

$$e^{-j\theta^0} x(n) + d_1^0 x(n-1) + \cdots + d_1^{0*} x(n-p+1) + e^{j\theta^0} x(n-p) = 0. \tag{4}$$

If we now apply the filter $G_p^0(z)$ to the noisy data then an error sequence is generated, i.e.,

$$e^{-j\theta^0} x(n) + d_1^0 x(n-1) + \cdots + d_1^{0*} x(n-p+1) + e^{j\theta^0} x(n-p) = e(n). \tag{5}$$

Notice that (5) denotes some special form of prediction where, using $x(n-1), \cdots, x(n-p+1)$, we predict in the forward and backward direction a combination of $x(n)$ and $x(n-p)$. It is exactly this predictor we are going to call the *complex bidirectional predictor* (CBP), which constitutes a generalization of the bidirectional predictor introduced in [5] for the real data case.

Our goal is, using the available data samples, to estimate a CBP of order $M+1$ and then by rooting the corresponding polynomial to estimate the unknown frequencies f_k, $k = 1, \cdots, p$. We must select $M \geq p$ in order to be able to identify correctly all frequencies (actually we must select $M \gg p$ to have a good estimation). Let $\boldsymbol{g}_{M+1}$ be the estimated CBP, which is of the form

$$\boldsymbol{g}_{M+1} = [e^{-j\theta} \boldsymbol{d}_{M-1}^t e^{j\theta}]^t = [e^{-j\theta} d_1 d_2 \cdots d_2^* d_1^* e^{j\theta}]^t \tag{6}$$

where $\boldsymbol{g}_{M+1}$, $\boldsymbol{d}_{M-1}$are again conjugate symmetric vectors. Thus, due to the conjugate symmetry property of the vector $\boldsymbol{g}_{M+1}$ a total of $\lceil \frac{M+1}{2} \rceil$ parameters must be estimated. Notice that the parameter θ is of a special form and we need to distinguish it from the other parameters in $\boldsymbol{d}_{M-1}$. If we now use the LS criterion to estimate $\boldsymbol{d}_{M-1}$ and θ then we can easily show that we must minimize the following cost function with respect to $\boldsymbol{d}_{M-1}$ and θ.

$$\mathcal{E}(\boldsymbol{d}_{M-1}, \theta) = \boldsymbol{g}_{M+1}^H Q_{M+1} \boldsymbol{g}_{M+1} \tag{7}$$

where the matrix Q_{M+1} is the associated data autocorrelation matrix defined as

$$\begin{aligned} Q_{M+1} &= R_{M+1} + J R_{M+1}^* J \\ R_{M+1} &= \sum_{n=M+1}^{N} \boldsymbol{x}_{M+1}^*(n) \boldsymbol{x}_{M+1}^t(n) \\ \boldsymbol{x}_{M+1}(n) &= [x(n) \cdots x(n-M)]^t \end{aligned} \tag{8}$$

and J is the so-called exchange matrix with unities along the cross diagonal and zeros everywhere else. The matrix Q_{M+1} is positive definite, Hermitian and persymmetric, i.e.,

$$Q_{M+1} = Q_{M+1}^H \quad Q_{M+1}^t = J Q_{M+1} J. \tag{9}$$

Let us now define a partition of Q_{M+1} that will be used next

$$Q_{M+1} = \begin{bmatrix} q & \boldsymbol{q}_{M-1}^H & s \\ \boldsymbol{q}_{M-1} & \tilde{Q}_{M-1} & J\boldsymbol{q}_{M-1}^* \\ s^* & \boldsymbol{q}_{M-1}^t J & q \end{bmatrix}. \tag{10}$$

Notice that the central matrix $\tilde{Q}_{M-1}$ is again a positive definite, Hermitian and persymmetric matrix. It can be easily seen that it is just a delayed version of Q_{M-1}. To obtain the optimum parameters we shall perform the minimization of (7) in two consecutive steps. First, the cost function $\mathcal{E}(\boldsymbol{d}_{M-1}, \theta)$ is minimized with respect to $\boldsymbol{d}_{M-1}$ assuming θ given. This results in the following optimum solution for $\boldsymbol{d}_{M-1}$

$$\boldsymbol{d}_{M-1} = J\boldsymbol{p}_{M-1}^* e^{-j\theta} + \boldsymbol{p}_{M-1} e^{j\theta} \tag{11}$$

where

$$\boldsymbol{p}_{M-1} = -\tilde{Q}_{M-1}^{-1} J \boldsymbol{q}_{M-1}^*. \tag{12}$$

The resulting value of the minimized cost function is given by

$$\mathcal{E}(\theta) = \min_{\boldsymbol{d}_{M-1}} \mathcal{E}(\boldsymbol{d}_{M-1}, \theta) = 2\mathrm{Re}\{a_{M+1} e^{j\theta}\} \tag{13}$$

where

$$a_{M+1} = q e^{-j\theta} + s e^{j\theta} + \boldsymbol{q}_{M-1}^H \boldsymbol{d}_{M-1}. \tag{14}$$

In the second step $\mathcal{E}(\theta)$ is minimized with respect to θ. After some algebra it can be shown that the optimum θ is given by

$$\theta = 0.5\left(\pi - \arg\{s + \boldsymbol{q}_{M-1}^H \boldsymbol{p}_{M-1}\}\right). \tag{15}$$

Equations (11) and (15) constitute the complete solution to the problem of LS estimating the CBP.

As it is already mentioned in the previous section, until now the only well known symmetric predictor applicable to the problem of spectral line estimation is the *complex smoother predictor* (CSP) [1], [4], [10], which estimates in the LS sense the sample $x(n)$ using m past and m future samples. The CSP is a conjugate symmetric predictor of only odd length that has the form

$$\boldsymbol{s}_{2m+1} = \begin{bmatrix} \boldsymbol{c}_m \\ 1 \\ J\boldsymbol{c}_m^* \end{bmatrix}. \tag{16}$$

To obtain the optimum in the LS sense CSP, the following cost function is minimized with respect to $\boldsymbol{s}_{2m+1}$.

$$\mathcal{E}(\boldsymbol{s}_{2m+1}) = \boldsymbol{s}_{2m+1}^H Q_{2m+1} \boldsymbol{s}_{2m+1}. \tag{17}$$

It can be readily shown [1], that the optimum CSP is the solution of the following linear system of equations

$$\hat{Q}_{2m \times (2m+1)} \boldsymbol{s}_{2m+1} = \boldsymbol{0}_{2m} \tag{18}$$

where $\hat{Q}_{2m \times (2m+1)}$ is the matrix Q_{2m+1} without its central row. Now, our final goal, which is the estimation of the unknown frequencies, can be achieved by rooting the polynomial formed by either of the above presented predictors (CBP or CSP).

Comment: It is interesting to notice that if we apply the CBP method in real data then we do not always obtain the real symmetric bidirectional predictor of [5]. Depending on the case, the method can also yield the optimum antisymmetric bidirectional predictor. In other words, the CBP cannot be considered as the complex case counterpart of the predictor of [5].

III. Asymptotic Characteristics of CBP And CSP

It is very difficult in general to compute analytically a performance measure for the two frequency estimators in the finite data case. In order to be able to derive some semi-analytic results we will consider the asymptotic (infinite data number) case. Specifically we will try to obtain expressions for the asymptotic bias.

A. Asymptotic Bias

Both predictors are biased even for the asymptotic case. Actually this is a fact for all known linear LS predictors applied to the same problem. Since the CSP can only be defined for odd orders, we shall compute the bias for this case only, that is, we will consider $M+1=2m+1$. As we have seen in the previous section the two predictors are defined through similar optimization problems. Let us see what is the form of these problems in the asymptotic case. Notice that the optimum estimated quantities remain the same if instead of the matrix Q we use the normalized matrix Q/N, where N is the number of available data samples. Letting $N \to \infty$ the autocorrelation matrix takes the form

$$Q_{2m+1} = \sigma_w^2 I_{2m+1} + V_{(2m+1)\times p} V_{(2m+1)\times p}^H \tag{19}$$

where σ_w^2 is the noise variance and

$$V_{(2m+1)\times p} = \begin{bmatrix} e^{-jm2\pi f_1} & e^{-jm2\pi f_2} & \cdots & e^{-jm2\pi f_p} \\ e^{-j(m-1)2\pi f_1} & e^{-j(m-1)2\pi f_2} & \cdots & e^{-j(m-1)2\pi f_p} \\ \vdots & \vdots & \cdots & \vdots \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \cdots & \vdots \\ e^{j(m-1)2\pi f_1} & e^{j(m-1)2\pi f_2} & \cdots & e^{j(m-1)2\pi f_p} \\ e^{jm2\pi f_1} & e^{jm2\pi f_2} & \cdots & e^{jm2\pi f_p} \end{bmatrix}. \tag{20}$$

We also have assumed that all complex exponentials that constitute the signal $x(n)$ have unit amplitude.

To find the optimum predictors for the asymptotic case we follow exactly the same steps as in the finite data case, that is (11) and (15) for the CBP and (18) for the CSP. For CBP, (11) after some algebraic manipulations and using the matrix inversion lemma yields the following asymptotic expression for $\boldsymbol{d}_{2m-1}$ given θ

$$\boldsymbol{d}_{2m-1} = -2V_{(2m-1)\times p}\left(\sigma_w^2 I_p + V_{(2m-1)\times p}^{\boldsymbol{H}} V_{(2m-1)\times p}\right)^{-1} \cdot \begin{bmatrix} \mathrm{Re}\{e^{j(2\pi f_1 m-\theta)}\} \\ \vdots \\ \mathrm{Re}\{e^{j(2\pi f_p m-\theta)}\} \end{bmatrix}. \tag{21}$$

In the second step an asymptotic expression for the phase θ is derived. Starting from the definition of θ in (15) and using the above results we obtain

$$\theta = \frac{\pi - \phi}{2} \tag{22}$$

where ϕ is defined

$$\phi = \arg\left\{ \left[e^{-j2\pi f_1 m} \cdots e^{-j2\pi f_p m}\right] \cdot \left(\sigma_w^2 I + V_{(2m-1)\times p}^{\boldsymbol{H}} V_{(2m-1)\times 2}\right)^{-1} \begin{bmatrix} e^{-j2\pi f_1 m} \\ \vdots \\ e^{-j2\pi f_p m} \end{bmatrix} \right\} \tag{23}$$

Thus, (21) and (22) give the asymptotic values of the bidirectional predictor coefficients.

In a similar manner it can be shown that the corresponding expression that yields the asymptotic values of the CSP is the following

$$\begin{bmatrix} \boldsymbol{c}_m \\ J\boldsymbol{c}_m^* \end{bmatrix} = -U_{2m\times p}\left(\sigma_w^2 I_p + U_{2m\times p}^{\boldsymbol{H}} U_{2m\times p}\right)^{-1} \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} \tag{24}$$

where

$$U_{2m\times p} = \begin{bmatrix} e^{-jm2\pi f_1} & e^{-jm2\pi f_2} & \cdots & e^{-jm2\pi f_p} \\ \vdots & \vdots & \cdots & \vdots \\ e^{-j2\pi f_1} & e^{-j2\pi f_2} & \cdots & e^{-j2\pi f_p} \\ e^{j2\pi f_1} & e^{j2\pi f_2} & \cdots & e^{j2\pi f_p} \\ \vdots & \vdots & \cdots & \vdots \\ e^{jm2\pi f_1} & e^{jm2\pi f_2} & \cdots & e^{jm2\pi f_p} \end{bmatrix}. \tag{25}$$

The obtained predictors are functions of the frequencies f_k, $k=1,\cdots,p$ and the noise variance σ_w^2. Using these formulae for the predictors we can form the corresponding polynomials and thus by rooting them obtain estimates of the frequencies for both predictors. Unfortunately it is not possible to derive closed form solutions for the estimated frequencies except for very small orders m. In any case this is not so important since we can always find numerically, for an adequate number of frequency combinations and variances, the asymptotic estimates and compare the two methods. For example in the case of a process consisting of two exponentials in noise a 3-D plot is required to illustrate the performance of a method (for given σ_w^2). However in the Appendix we show that the asymptotic bias is the same (in absolute value) for both estimated frequencies and that depends only on the difference $\Delta f = |f_1 - f_2|$. This statement is true for both methods. Thus we can use a 2-D plot in order to illustrate the performance of both methods. Specifically we can plot the absolute bias with respect to Δf for the two predictors. Fig. 1 depicts such an example. The solid line corresponds to the CBP and the dashed to the CSP. We used $m=6$ (predictor order equal to 13) for both cases and $\sigma_w^2 = 0.01$. Clearly the CBP compared to the CSP exhibits globally a much smaller bias. The results for other orders and other values of the noise variance are similar.

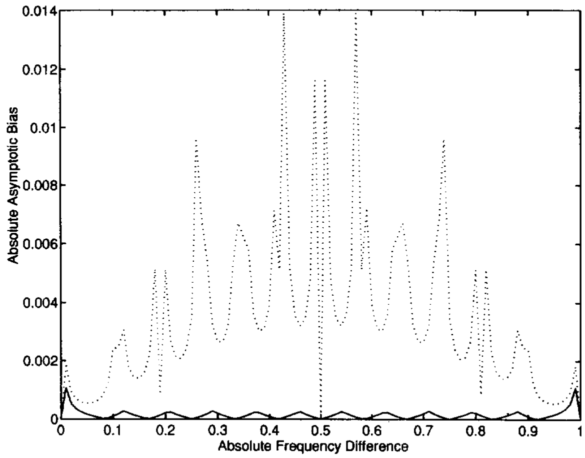


Fig. 1. Asymptotic bias for LS optimum CBP (solid) and CSP (dotted) for predictor order equal to 13 and SNR = 20 dB.

B. Asymptotic Variance

A very important performance measure is the estimation error variance. Analytic expressions for this case are very difficult even for the asymptotic case. However since this measure is very important we shall investigate this matter in Section V through extensive simulations, as is usually the case in the literature.

IV. Given Data Case and Computational Issues

Frequency estimation using linear prediction techniques involves usually three separate steps. The first step consists in identifying, using the available data, a linear prediction model. In the second step the roots of the polynomial formed by the linear model parameters are computed. The desired frequency estimates constitute a subset of the angles of the obtained roots. In the final step the selection of the desired subset is performed.

In this paper, we are concerned only with the first two steps. For the third step there exist several efficient methods in the literature [6], [12], [13]. All these methods exploit properly the special structure of the associated problem and exhibit a computational complexity that is problem dependent and in most cases, does not exceed $O(M^2)$.

Next, we examine computational issues concerning the first two steps when the CBP predictor is adopted for the frequency estimation problem.

A. Efficient LS Solutions

The computation of CBP (or CSP) via the conventional solution of the respective linear systems of equations requires $O(M^3)$ operations. Taking into account the rich structure of these systems and developing order recursive (Levinson-type) algorithms the above computational burden can be reduced to $O(M^2)$. In this paper however we will not derive these algorithms. The reason is that, as we explain next, they can be obtained by properly extending an existing algorithm established for the FBLP problem.

In the case of CBP, it is evident from (11) and (15) that the problem of computing $\boldsymbol{g}_{M+1}$ is reduced to the computation of the auxiliary vector $\boldsymbol{p}_{M-1}$. An interesting fact is that this vector is a by-product of the fast covariance algorithm of complexity $O(M^2)$ derived in pp. 251–257 of [1] and used for the estimation of the FBLP. In this algorithm, the vector $\boldsymbol{p}_{M-1}$ appears as an auxiliary predictor in the algorithmic process. Thus, the efficient algorithm in [1] can be used as it is for the computation of $\boldsymbol{p}_{M-1}$. The inner product of (15) is also computed in the same algorithm thus the only extra operations are those required for (11).

Notice that an efficient order recursive algorithm for the estimation of the CSP is also available in [1]. This algorithm is again based on the algorithm used for the FBLP and by embedding some additional recursive relations succeeds in obtaining CSP. Thus, the CSP can also be derived as a by-product of the same order recursive algorithm used for the CBP, requiring some small extra computational cost.

Concluding, the model identification algorithm for FBLP, CSP and CBP has a complexity that is $O(M^2)$.

B. Location of the Roots

The second step in the process of estimating undamped exponentials through the use of predictors is the determination of the roots of the corresponding polynomials. As we will next see, this is the most time consuming part for LS methods. It is clear that any information facilitating the determination of the roots can lead to an improvement of the overall complexity.

We will now show that the CBP predictor has all its roots on the unit circle whenever the FBLP of the corresponding problem is stable. Since the FBLP is asymptotically stable we conclude that the CBP has all its roots on the unit circle in the asymptotic case. Furthermore the FBLP preserves its very good stability properties in the finite data case for well conditioned matrices Q_{M+1}. This means that the CBP will have, in almost all cases, its roots on the unit circle. This fact was also certified through extensive simulations as we will see in the next section.

Let us now define the FBLP and show its connection to the CBP. The FBLP is defined as the solution to the following linear problem

$$Q_{M+1}\begin{bmatrix}1\\ \boldsymbol{a}_M\end{bmatrix} = \begin{bmatrix}\alpha_M\\ \boldsymbol{o}_M\end{bmatrix} \tag{26}$$

where α_M is the prediction error power and thus is a real number. If β is now any complex number, then we have that

$$Q_{M+1}\left\{\beta\begin{bmatrix}1\\ \boldsymbol{a}_M\end{bmatrix} + \beta^*\begin{bmatrix}J\boldsymbol{a}_M^*\\ 1\end{bmatrix}\right\} = \begin{bmatrix}\beta\alpha_M\\ \boldsymbol{0}_{M-1}\\ \beta^*\alpha_M\end{bmatrix}. \tag{27}$$

Notice that the vector in the brackets is conjugate symmetric. It can also be easily proved that it is always possible to select β so that its first element is equal to $e^{-j\theta}$, with θ defined in (15). If β is selected this way then the vector

$$\boldsymbol{g}_{M+1} = \beta\begin{bmatrix}1\\ \boldsymbol{a}_M\end{bmatrix} + \beta^*\begin{bmatrix}J\boldsymbol{a}_M^*\\ 1\end{bmatrix} \tag{28}$$

can be shown to be the CBP by verifying that (27) is the augmented system of normal equations for the CBP, i.e., that (11) and (15) are satisfied by $\boldsymbol{g}_{M+1}$ of (28). Since our predictor can be written under the form of (28), then extending the result of [9] to the complex polynomial case, we conclude that the CBP will have all its roots on the unit circle whenever the FBLP is stable. As we said this is asymptotically true and it is true most of the time for well conditioned matrices Q_{M+1}. We must however stress at this point that requiring FBLP to be stable is only a SUFFICIENT condition for CBP to have its roots on the unit circle. In practice we have observed that this property appears to hold for CBP more often than the stability of the corresponding FBLP. Notice also that this property for the location of the polynomial roots does not hold even asymptotically for the CSP case as one can verify by a counterexample (take for instance the case $M = 2$ that can be solved analytically).

Since a general rooting procedure has complexity $O(M^3)$ [14] it is clear that the rooting part is the most computationally heavy part in the whole frequency estimation problem (for LS based techniques). Because of the characteristic root location property of the CBP a significant computational saving can be achieved in the rooting procedure. Two possible schemes are presented next.

C. Rooting Scheme Based on Searching on the Unit Circle

Although in the finite data case the CBP does not always has all its roots on the unit circle the percentage of the non unit modulus roots is very small as was confirmed by extensive simulations (thousands of independent experiments). Specifically, when $M + 1 \leq N/2$ (N being the number of samples) the percentage of the roots with non unit modulus varies from 0% at SNR = 30 dB to 2% at SNR = 10 dB. Additionally, the modulus of these roots is always very close to unity. The same experiments for CSP yield quite different results. Roots of non unit modulus occur more often at a rate of 30 to 90%, depending on the SNR. In addition, their modulus, most of the time, is significantly different from unity.

We thus conclude that if we incorporate for CBP a root finder that searches only on the unit circle the reduction in performance (as compared to using general root finders) will be insignificant. On the other hand, root finders searching only on the unit circle exhibit complexity $O(M^2)$ [8], [11], which is an order of magnitude smaller than the complexity of general rooting schemes. Note that under such a rooting scheme the complexity of the second step (rooting) becomes of the same order as the other two steps (identification and frequency selection).

D. Rooting Scheme Involving Real Polynomials

There also exists an alternative scheme that one can incorporate for the rooting of conjugate symmetric polynomials in general. Thus the proposed method can be also used for CSP. Notice though that this method does not reduce the complexity by an order of magnitude but rather by a constant percentage.

One can easily prove that in a conjugate symmetric polynomial the roots lie either on the unit circle or appear in mirror, with respect to the unit circle, pairs. Thus, if the original polynomial is $G(z)$ and we apply a change of variables using the bilinear transformation

$$z = \frac{1 + js}{1 - js} \tag{29}$$

we can obtain a new polynomial $G'(s)$ that has real coefficients. This is so because under this transformation, roots located on the unit circle are mapped to real roots, whereas mirror pairs to complex conjugate pairs. Thus, the resulting polynomial has only real and/or complex conjugate roots, meaning that either it is or it can be (trivially) reduced to a real polynomial. The number of operations needed to obtain the new polynomial $G'(s)$ from $G(z)$ is $O(M^2)$.

Most widely used rooting algorithms that obtain the roots one by one, have exactly the same form for both real and complex polynomials [18]. In addition, at every step these algorithms perform a number of polynomial evaluations which constitutes the main bulk of operations. The polynomials involved are usually the original and derivatives of the original polynomial. This means that if the original polynomial is real, so is its derivative and all higher derivatives, which implies that every step of the method involves evaluation of only real polynomials. If we are thus searching for real roots we will have only real operations in the evaluation process. For the evaluation of polynomials, Horner's scheme is usually used requiring M multiplications and M additions [15]. When these operations are between complex numbers then this corresponds to $4M$ real multiplications and $4M$ real additions. Thus it is clear that when we evaluate real polynomials with real arguments then this requires 75% less operations than evaluating complex polynomials with complex arguments. On the other hand if we are looking for a complex root, even if the polynomial is real, there is absolutely no gain in the evaluation process with Horner's scheme (i.e., requires the same number of operations as a complex polynomial). Since for the CBP the (transformed) roots are mostly real, we will have this 75% reduction of the computation. In other words, the complexity will be one forth of the complexity required for the determination of complex roots. For CSP the gain depends on the number of real and complex roots that appear in the corresponding polynomial.

In any case, if the polynomial is real it is possible to apply other techniques that involve only real arithmetic. For example instead of finding the roots one by one, we can look for real second-order polynomials that divide the original polynomial. This corresponds to simultaneously obtaining complex conjugate root pairs [15].

V. Performance Comparison of LS Predictors

Thus far, we have seen, through a numerical example, that the CBP exhibits less asymptotic bias. In order to test the proposed method in the finite data case we will conduct the typical experiment of [6]. Specifically we will apply the methods we like to compare to a number of independent data realizations so as to derive conclusions about the statistical behavior of the estimation error. The performance of the methods will be tested through the mean estimation error (bias)

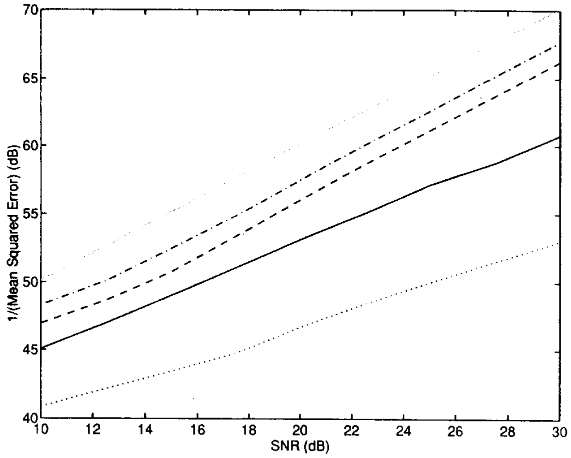


Fig. 2. Estimation error variance for LS-based methods: CBP (solid), FBLP (dashed), CSP (dotted), combination CBP-FBLP (dash-dotted), and Cramer–Rao bound (half-tone).

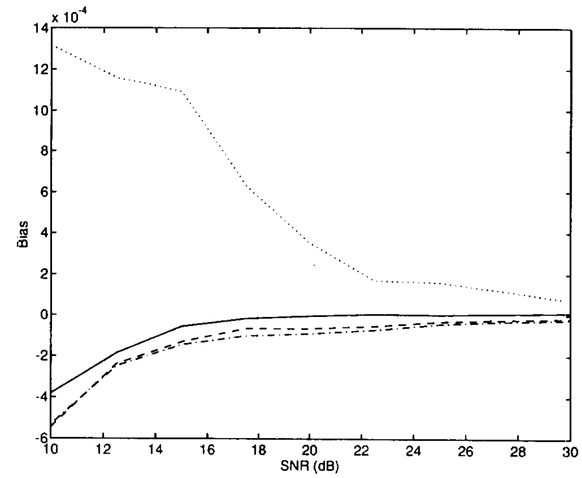


Fig. 3. Estimation error bias for LS-based methods: CBP (solid), FBLP (dashed), CSP (dotted), combination CBP-FBLP (dash-dotted).

and the estimation error variance. For this reason a block of 25 signal points is generated using the formula

$$s(n) = e^{j2\pi 0.5n} + e^{j(2\pi 0.52n + \pi/4)} \quad n = 0, 1, \cdots, 24. \quad (30)$$

To the signal block we add 500 statistically independent noise blocks. Each noise block contains 25 samples of complex white Gaussian noise with variance σ_w^2. The SNR per exponential is defined as $10\log_{10}(1/2\sigma_w^2)$ and SNR values range from 10 to 30 dB. For each block of data, each SNR value and each estimation method, the angles of the roots closest to the true frequencies, are considered as the frequency estimates. This is of course an ideal case but it is indicative of the true performance of each method. The bias and the variance σ_f^2 of the frequency estimate of f_1 is computed over the 500 realizations for each SNR value. In Fig. 2, the quantity $10\log_{10}(1/\sigma_f^2)$ is plotted versus SNR, for the different methods. Fig. 3 presents the corresponding biases. For each particular method the order that had the smallest variance was selected. In both figures the solid line corresponds to the CBP with order equal to 12, the dashed line to the FBLP with order equal to 13 and the dotted line to the CSP with order 13. We also present in Fig. 2 the Cramer–Rao bound (half-tone line). The estimation variance of the CBP is 2–5 dB inferior to that of the FBLP, whereas the CSP has a significantly poorer performance. On the other hand, the bias of the CBP is the smallest among all methods.

A. Combination of the CBP and FBLP

From the above we conclude that the FBLP method is a few decibels superior in performance with respect to the CBP method. On the other hand, the CBP method is computationally the most efficient among the existing LS methods since it reduces drastically the complexity of the rooting part. In the following, we briefly describe a method which performs like the FBLP method (and even better) and has a computational complexity practically the same with that of the CBP method.

Recall that both predictors (FBLP and CBP) are obtained as by-products of the same algorithm as discussed in Section IV. Thus, for each block of data, both predictors are computed with almost no additional complexity using basically the algorithm of [1]. This is the first step of the combination method. In the second step the CBP is rooted. From its roots, we select the ones that can be considered as estimates of the true frequencies. These roots are in turn used as starting points to any appropriate rooting algorithm in order to compute the corresponding zeros of the FBLP polynomial. Combining the angles of the two sets of roots, we select the angles that are closest to the true frequencies as our final estimates. This process might yield selections from either the CBP roots or the FBLP roots or both.

In our experiments, we used the Muller scheme [15] to obtain the FBLP roots from the CPB roots. In all the experiments, less than 10 iterations were enough for the scheme to converge. Note that the complexity of the Muller scheme is approximately $2M$ operations per iteration. Thus, the number of extra operations needed to obtain the required subset of FBLP roots is $O(M^2)$, yielding a total complexity of the same order. This complexity is an order of magnitude smaller than the one used in the FBLP method ($O(M^3)$) and required by the general rooting algorithm.

The dashed-dotted line in Figs. 2 and 3 corresponds to the above described combination of the CBP and FBLP. As we can observe the combination method exhibits a smaller than the FBPL variance, whereas the bias is almost the same.

Remark: It should be stressed again that in the simulations the frequencies closest to the true frequencies were used as our estimates. This was done for simplicity and in order to observe the true capabilities of each method. In an actual case however the true frequencies are unknown and the rooting part must be accompanied by a root selection method. A variety of methods based on different criteria have been proposed so far [6], [12], [13]. All these methods were tested in our case and the one in [12] was found to be appropriate for the proposed techniques, mainly for two reasons. First, it exhibits very low thresholds, and second, it does not use as criterion the closeness of the roots to the unit circle (recall that all roots of the bidirectional polynomial lie on the unit circle). We applied this method in all the above experiments and the performance of all methods, under this selection technique, was very similar to the ideal one.

VI. Minimum Norm Estimation of CBP and CSP Based on SVD Decomposition

In this section, a minimum norm method will be presented for obtaining the CBP and the CSP. The method is based on SVD decomposition of the data correlation matrix and follows similar steps as in [7]. Methods that are based on SVD are known to yield superior results compared to their LS counterparts because, with the use of SVD, the effect of noise on the data is drastically reduced. As we will see in the simulation results, this is also the case for the CBP and CSP. Specifically, both predictors improve their performance by almost 5–10 dB and extend their range of acceptable performance significantly.

Before defining the optimization problems that will lead us to the optimum symmetric predictors, let us first explore the rich structure of the data correlation matrix Q_{M+1}. As we have seen in Section II, this matrix is positive definite, Hermitian, and persymmetric satisfying (9). We can thus prove the following lemma.

Lemma: All eigenvectors of Q_{M+1} can take a conjugate symmetric form.

Proof: Since the matrix Q_{M+1} is Hermitian, it has real eigenvalues (actually positive). Thus, if $\lambda, \boldsymbol{v}$ is an eigenvalue-eigenvector pair, then using (9), we have

$$\begin{aligned} Q_{M+1}J\boldsymbol{v}^* &= JQ_{M+1}^t\boldsymbol{v}^* = JQ_{M+1}^*\boldsymbol{v}^* \\ &= J(Q_{M+1}\boldsymbol{v})^* = \lambda J\boldsymbol{v}^* \end{aligned} \tag{31}$$

which suggests that $J\boldsymbol{v}^*$ is also an eigenvector for λ. If $\boldsymbol{v}$, $J\boldsymbol{v}^*$ are eigenvectors for λ, so is their sum $\boldsymbol{v} + J\boldsymbol{v}^*$, which is conjugate symmetric. This completes the proof.

There are two reasons for introducing conjugate symmetric eigenvectors. First, notice that both predictors of our interest have this conjugate symmetric form and thus can be easily defined as a linear combination of conjugate symmetric eigenvectors. The second reason is that, as we will prove in the next theorem, conjugate symmetric eigenvectors can be obtained through an eigendecomposition problem of a real symmetric matrix. It is clear that for such a case the necessary computational complexity is lower as compared to the complex case.

Theorem: If $\lambda, \boldsymbol{v}$ is an eigenvalue-eigenvector pair for the matrix Q_{M+1} with $\boldsymbol{v}$ a conjugate symmetric vector, then this pair can be obtained by the following eigendecomposition problem.

Case $M+1=2m$:

$$\lambda\begin{bmatrix}\boldsymbol{v}_r\\ \boldsymbol{v}_i\end{bmatrix} = \begin{bmatrix}A_r+B_rJ & -A_i+B_iJ\\ A_i+B_iJ & A_r-B_rJ\end{bmatrix}\begin{bmatrix}\boldsymbol{v}_r\\ \boldsymbol{v}_i\end{bmatrix} \tag{32}$$

where the matrix Q_{2m} and the eigenvector $\boldsymbol{v}$ have the form

$$\begin{aligned} Q_{2m} &= \begin{bmatrix} A_r+jA_i & B_r+jB_i\\ JB_rJ-jJB_iJ & JA_rJ-jJA_iJ\end{bmatrix}\\ \boldsymbol{v} &= \begin{bmatrix}\boldsymbol{v}_r+j\boldsymbol{v}_i\\ J\boldsymbol{v}_r-jJ\boldsymbol{v}_i\end{bmatrix}. \end{aligned} \tag{33}$$

Case $M+1=2m+1$:

$$\lambda\begin{bmatrix}\boldsymbol{v}_r\\ \xi/\sqrt{2}\\ \boldsymbol{v}_i\end{bmatrix} = \begin{bmatrix}A_r+B_rJ & \sqrt{2}\boldsymbol{b}_r & -A_i+B_iJ\\ \sqrt{2}\boldsymbol{b}_r^t & \omega/\sqrt{2} & \sqrt{2}\boldsymbol{b}_i^t\\ A_i+B_iJ & \sqrt{2}\boldsymbol{b}_i & A_r-B_rJ\end{bmatrix}\begin{bmatrix}\boldsymbol{v}_r\\ \xi/\sqrt{2}\\ \boldsymbol{v}_i\end{bmatrix} \tag{34}$$

where the matrix Q_{2m+1} and the eigenvector $\boldsymbol{v}$ have the form

$$\begin{aligned} Q_{2m+1} &= \begin{bmatrix}A_r+jA_i & \boldsymbol{b}_r+j\boldsymbol{b}_i & B_r+jB_i\\ \boldsymbol{b}_r^t-j\boldsymbol{b}_i^t & \omega & \boldsymbol{b}_r^tJ+\boldsymbol{b}_i^tJ\\ JB_rJ-jJB_iJ & J\boldsymbol{b}_r-jJ\boldsymbol{b}_i & JA_rJ-jJA_iJ\end{bmatrix}\\ \boldsymbol{v} &= \begin{bmatrix}\boldsymbol{v}_r+j\boldsymbol{v}_i\\ \xi\\ J\boldsymbol{v}_r-jJ\boldsymbol{v}_i\end{bmatrix} \end{aligned} \tag{35}$$

and both matrices involved in the two real eigendecomposition problems are symmetric.

Proof: The proof is given in the Appendix.

A. Definition of the Optimum Symmetric Predictors

Let E_s, E_n denote the collection of singular vectors (that coincide with the eigenvectors) that correspondingly span the signal and noise subspaces of the matrix Q_{M+1}. As we have seen in the theorem, these vectors can be obtained through a real eigendecomposition problem. Our aim now is to find a conjugate symmetric predictor of the form of the CBP or CSP that is orthogonal to the signal subspace E_s and has minimum norm. Depending on the predictor type, we have different constraints for the minimum norm problem. Specifically, for the CBP, we require the two end predictor elements to have unit magnitude, whereas for the CSP, the central element should be equal to unity.

Let us recall that the vectors in E_n are conjugate symmetric. Thus, if we like to form a conjugate symmetric vector by linearly combining the vectors of E_n we must use for the combination only REAL coefficients. In other words, the predictors we are looking for, can be put under the form $E_n\boldsymbol{t}_n$, where $\boldsymbol{t}_n$ is a real vector. Let us now consider the following two partitions of the matrix E_n

$$E_n = \begin{bmatrix}\boldsymbol{e}^t\\ E'\\ \boldsymbol{e}^H\end{bmatrix} \quad \text{and} \quad E_n = \begin{bmatrix}L\\ \boldsymbol{l}^t\\ JL^*\end{bmatrix} \tag{36}$$

where the $\boldsymbol{e}^t$ is the top row of the matrix E_n while the $\boldsymbol{l}^t$ is the central row. The first partition of E_n will be used for the CBP problem and the second for the CSP. For the second partition notice that it can be applied only to odd length vectors and since the vectors in E_n are conjugate symmetric we conclude that $\boldsymbol{l}$ is a real vector.

From the orthonormality of the singular vectors we conclude that $\|E_n \boldsymbol{t}_n\| = \|\boldsymbol{t}_n\|$. Thus, looking for the minimum norm combination is equivalent to looking for the minimum norm real vector $\boldsymbol{t}_n$. The solution to the minimum norm problem is given by the following proposition.

Proposition: Let us define a conjugate symmetric predictor as $E_n \boldsymbol{t}_n$. Then, the optimum vectors $\boldsymbol{t}_n$ that yield a minimum norm CBP and a minimum norm CSP are defined as the solution to the following optimization problems.

Case CBP: The solution to

$$\min_{\boldsymbol{t}_n} \|\boldsymbol{t}_n\|^2 \text{ under } |\boldsymbol{e}^t \boldsymbol{t}_n| = 1 \tag{37}$$

is given by

$$\boldsymbol{t}_n = [\boldsymbol{e}_r \boldsymbol{e}_i] \frac{\boldsymbol{x}_\lambda}{\lambda \|\boldsymbol{x}_\lambda\|} \tag{38}$$

where $\boldsymbol{e} = \boldsymbol{e}_r + j\boldsymbol{e}_i$ and $\lambda, \boldsymbol{x}_\lambda$ is the largest eigenvalue and the corresponding eigenvector of the following 2×2 matrix

$$\begin{bmatrix} \|\boldsymbol{e}_r\|^2 & \boldsymbol{e}_r^t \boldsymbol{e}_i \\ \boldsymbol{e}_i^t \boldsymbol{e}_r & \|\boldsymbol{e}_i\|^2 \end{bmatrix}. \tag{39}$$

Case CSP: The solution to

$$\min_{\boldsymbol{t}_n} \|\boldsymbol{t}_n\|^2 \text{ under } \boldsymbol{l}^t \boldsymbol{t}_n = 1 \tag{40}$$

is given by

$$\boldsymbol{t}_n = \frac{\boldsymbol{l}}{\|\boldsymbol{l}\|^2}. \tag{41}$$

Proof: The proof of the proposition is given in the Appendix.

B. Computational Issues

As in the LS methods, we will examine the complexities of the first two steps (identification and rooting) of the whole frequency estimation process for the above mentioned methods.

Note that for minimum norm FBLP, CBP, CSP as well as for Root-MUSIC, in order to identify the corresponding predictors, we need to perform SVD on the matrix Q_{M+1}. SVD requires complexity $O(M^3)$ (see p. 239 of [16]) regardless of the type of the matrix (real or complex) Q_{M+1}. Unfortunately, in the literature, it was not possible to find exact complexities for the two types so that we could compare them exactly. We instead performed simulations in MATLAB and using the command "flops" [17] we counted the operations required for each type. The conclusion was that for the real case it required approximately 50 to 70% less operations than in the complex case for matrix sizes ranging from $M = 1$ to $M = 200$. We must emphasize that this gain in SVD is possible for all the above methods since there is no problem in any of them if we require the singular vectors to be conjugate symmetric.

As was the case in the LS method, the minimum norm CBP has most of the time its roots on the unit circle. Actually the percentage of non unit modulus roots is much smaller than in the LS case. Specifically, in thousands of experiments, only 2% of the roots were found not lying on the unit circle at SNR=0 dB and practically no such root occurred above SNR=15 dB. This was true for predictor orders close to the value $3N/4$ (which is known from practice to yield the best results for other SVD methods [6]). For other orders, the percentage of nonunit modulus roots was a little higher.

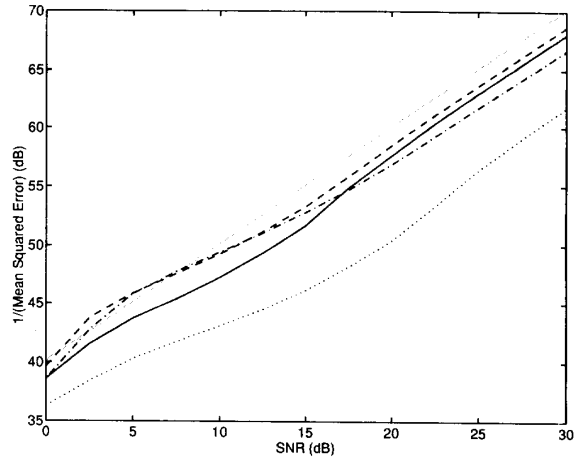


Fig. 4. Estimation error variance for SVD-based methods: CBP (solid), FBLP (dashed), CSP (dotted), Root-MUSIC (dash-dotted), and Cramer–Rao bound (half-tone).

Due to the root positioning in the case of the CBP, we can again apply either of the two methods presented in Section IV to reduce the complexity of the rooting part. Notice though that here, even if we reduce this complexity by an order of magnitude (using root searching on the unit circle), the overall complexity remains $O(M^3)$ because of SVD. In any case even if we have not gained an order of magnitude in the overall complexity we have nevertheless gained a significant saving in complexity. Note also that since CSP and Root-MUSIC require the rooting of complex conjugate polynomials it is possible to use for these two frequency estimators the method of rooting real polynomials of Section IV.

C. Simulation Results

To test the performance of the two new minimum norm predictors we conducted the same experiments as in Section V. The results are presented in Figs. 4 and 5. The solid line is the CBP with order 17, the dashed line is the minimum norm FBLP with order 19, the dotted line the CSP with order 17 and finally the dashed-dotted line the Root-MUSIC with order 19. We also present in Fig. 4 the Cramer–Rao bound (half-tone line). Again, we selected the orders for each case that yielded the smallest variance. Observing the estimation variance and the bias, the CBP behaves comparably to the minimum norm and Root-MUSIC while again the CSP has a significantly poorer performance. Notice that in this case it is not really necessary to define a combined CBP-FBLP scheme (as the one in the LS case) since it yields only an insignificant improvement.

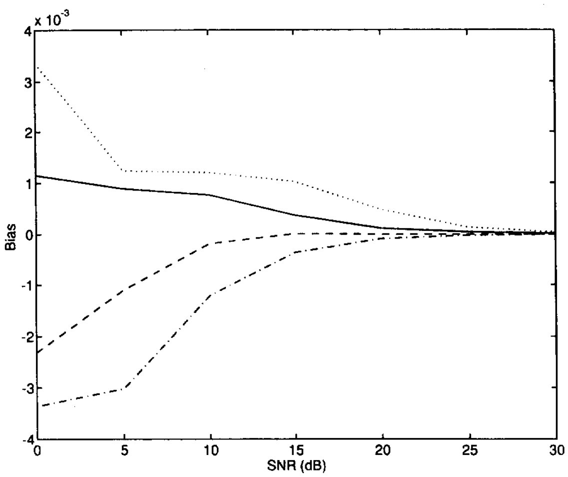


Fig. 5. Estimation error bias for SVD-based methods: CBP (solid), FBLP (dashed), CSP (dotted), Root-MUSIC (dash-dotted).

VII. Conclusion

A new conjugate symmetric predictor for estimating frequencies of sinusoids in noise was presented. The optimum form of the predictor was obtained by using the LS and the minimum norm criterion. The proposed predictor, in simulations, exhibited better performance than the well known symmetric Smoother predictor and comparable performance to the FBLP and Root-MUSIC. Also, the proposed predictor was shown to have all its roots on the unit circle in the asymptotic case, while in the finite data case, through extensive simulations, it exhibited this property more than 98% of the cases. This fact allows for the use of specialized rooting algorithms that have very low complexity, as compared to general rooting algorithms.

For the LS case adequate hints were given on how to derive a fast order recursive algorithm for the proposed predictor based on an existing algorithm for FBLP. A new technique combining the proposed predictor and the FBLP in the LS case was shown to have extremely good performance and low complexity. Finally, the eigenstructure of the Hermitian and persymmetric matrices was investigated and a useful theorem was proved. Based on this result a real SVD problem, necessary for the minimum norm FBLP, CBP, CSP methods, and the Root-MUSIC method was defined having reduced complexity as compared to the commonly used complex SVD problem.

Appendix A
Properties of CBP and CSP Asymptotic Bias

For the LS case, the asymptotic bias for CBP, CSP in estimating two frequencies f_1, f_2 is the same in absolute value for both frequencies; it also depends only on their difference $f_1 - f_2$.

We will show the above properties for CSP only; it is a little more involved for CBP. Using (24) for $p = 2$ and forming the corresponding polynomial, we obtain

$$G(z) = 1 + \rho \sum_{\substack{k=-m \\ k\neq 0}}^{m} \left(e^{-jk2\pi f_1} + e^{-jk2\pi f_2}\right) z^k \tag{42}$$

where

$$\rho^{-1} = \sigma_w^2 + 4\sum_{k=1}^{m} \cos^2\left(k2\pi\left(\frac{f_1 - f_2}{2}\right)\right). \tag{43}$$

If we change variables in the above polynomial as follows

$$z = y e^{j2\pi \frac{f_1+f_2}{2}} \tag{44}$$

we obtain the following polynomial in y

$$G'(y) = 1 + 2\rho \sum_{k=1}^{m} \cos\left(k\pi\left(\frac{f_1 - f_2}{2}\right)\right)\left(y^k + y^{-k}\right). \tag{45}$$

Notice that with the above change of variables, the two desired roots $e^{j2\pi f_1}$, $e^{j2\pi f_2}$ are transformed to $e^{j2\pi \frac{f_1-f_2}{2}}$, $e^{-j2\pi \frac{f_1-f_2}{2}}$, which is a reciprocal pair. We can thus conclude that the roots of $G'(y)$ try to estimate this pair. Note that the roots of $G'(y)$ come in reciprocal pairs as well. Thus, if $\mu e^{j2\pi\chi}$, $\mu^{-1}e^{-j2\pi\chi}$ is such a pair used for the estimation of the desired roots then the corresponding estimates for the two initial frequencies, using (44), will be

$$\hat{f}_1 = \frac{f_1 + f_2}{2} + \chi, \quad \hat{f}_2 = \frac{f_1 + f_2}{2} - \chi. \tag{46}$$

Forming the absolute error (absolute asymptotic bias), we can see that it is equal to $|\chi - (f_1 - f_2)/2|$ for both cases. Since χ comes from (45) it depends only on the difference of the two initial frequencies. We thus conclude that both absolute asymptotic biases are equal and depend only on the difference $f_1 - f_2$.

Appendix B
Proof of the Theorem

It can be easily proved using the persymmetry property that the two possible forms of Q_{M+1} (depending on M being even or odd) are the ones given in (33) and (35). From the Hermitian symmetry of Q_{M+1}, we have that

$$\begin{aligned} A_r^t &= A_r \quad A_i^t = -A_i \\ JB_r^t J &= B_r \quad JB_i^t J = B_i \end{aligned} \tag{47}$$

and that ω is real. Because of the conjugate symmetry of $\boldsymbol{v}$, the element ξ in (35) is also real. We will now prove the theorem for the first case only. The proof is similar for the other case. The eigendecomposition problem we are interested in is

$$\lambda \boldsymbol{v} = Q_{2m}\boldsymbol{v}. \tag{48}$$

Because of the Hermitian form of Q_{2k}, we know that all eigenvalues are real thus substituting Q_{2k} and $\boldsymbol{v}$ from (33) in (48) and separating real from imaginary parts, after performing block operations, we have

$$\begin{aligned} \lambda \boldsymbol{v}_r &= (A_r + B_r J)\boldsymbol{v}_r + (-A_i + B_i J)\boldsymbol{v}_i \\ \lambda \boldsymbol{v}_i &= (A_i + B_i J)\boldsymbol{v}_r + (A_r - B_r J)\boldsymbol{v}_i \end{aligned}. \tag{49}$$

Actually there are two additional relations, but they are equivalent to the ones given in (49). Notice that (32) is exactly (49) in matrix form. Using (47), it is easy to see that the matrix involved in (32) is symmetric. This concludes the proof.

Appendix C
Proof of the Proposition

We shall prove the proposition for the CBP case only, it can be similarly proved for the other case. Notice that, since $\boldsymbol{t}_n$ is real, we can write for the constraint

$$|\boldsymbol{e}^t \boldsymbol{t}_n|^2 = (\boldsymbol{e}_r^t \boldsymbol{t}_n)^2 + (\boldsymbol{e}_i^t \boldsymbol{t}_n)^2 = \boldsymbol{t}_n^t [\boldsymbol{e}_r \boldsymbol{e}_i][\boldsymbol{e}_r \boldsymbol{e}_i]^t \boldsymbol{t}_n = 1. \quad (50)$$

It is known that for any symmetric matrix T and any vector $\boldsymbol{t}$ we have

$$\frac{\boldsymbol{t}^t T \boldsymbol{t}}{\boldsymbol{t}^t \boldsymbol{t}} \leq \lambda_{\max}(T) \quad (51)$$

where $\lambda_{\max}(T)$ is the largest eigenvalue of T. We have equality in (51) when $\boldsymbol{t}$ is equal to the eigenvector that corresponds to $\lambda_{\max}$. Using (50) and (51), we conclude that

$$\|\boldsymbol{t}_n\|^2 \geq 1/\lambda_{\max}([\boldsymbol{e}_r \boldsymbol{e}_i][\boldsymbol{e}_r \boldsymbol{e}_i]^t) \quad (52)$$

with equality when $\boldsymbol{t}_n$ is the eigenvector corresponding to $\lambda_{\max}$. It is known that for any matrix T if λ, $\boldsymbol{x}_\lambda$ is an eigenvalue-eigenvector pair for $T^t T$ then λ, $T\boldsymbol{x}_\lambda$ is an eigenvalue–eigenvector pair for TT^t. Because of this statement we conclude that $\boldsymbol{t}_n$ can take the form defined in the proposition. The multiplicative constant in the definition of $\boldsymbol{t}_n$ in (38) is needed in order for $\boldsymbol{t}_n$ to meet the constraint. This concludes the proof.

References

[1] S. L. Marple, *Digital Spectral Analysis with Applications.* Englewood Cliffs, NJ: Prentice Hall, 1987.

[2] S. Haykin, *Adaptive Filter Theory.* Englewood Cliffs, NJ: Prentice Hall, 1991.

[3] S. M. Kay, *Modern Spectral Estimation: Theory and Applications.* Englewood Cliffs, NJ: Prentice Hall, 1988.

[4] M. Amin and L. J. Griffiths, "Time varying spectral estimation using symmetric smoothing," in *Proc. ICASSP-83* (Boston, MA), 1983, pp. 9–12.

[5] K. Berberidis and S. Theodoridis, "New Levinson, Schur, and lattice type algorithms for linear phase filtering," *IEEE Trans. Acoustic Speech Signal Processing*, vol. 38, pp. 1879–1892, Nov. 1990.

[6] D. W. Tufts and R. Kumaresan, "Estimation of frequencies of multiple sinusoids: making linear prediction perform like maximum likelihood," *Proc. IEEE*, vol. 70, pp. 975–989, Sept. 1982.

[7] R. Kumaresan and D. W. Tufts, "Estimating the angles of arrival of multiple plane waves," *IEEE Trans. Aerospace Electron. Syst.*, vol. AES-19, pp. 134–139, June 1983.

[8] W. S. McCormick and J. L. Lansford, "Efficient parallel rooting of complex polynomials on the unit circle," *IEEE Trans. Signal Processing*, vol. 39, pp. 2347–2351, Oct. 1991.

[9] A. Lepschy, G. A. Mian, and U. Viaro, "A note on line spectral frequencies," *IEEE Trans. Acoustic Speech Signal Processing*, vol. 36, pp. 1355–1357, Aug. 1988.

[10] S. L. Marple, "Fast algorithms for linear prediction and system identification filters with linear phase," *IEEE Trans. Acoustic Speech Signal Processing*, vol. ASSP-30, pp. 942–953, Dec. 1982.

[11] A. H. Nuttall, "Efficient Evaluation of polynomials and exponentials of polynomials for equispaced arguments," *IEEE Trans. Acoustic Speech Signal Processing*, vol. ASSP-35, pp. 1486–1487, Oct. 1987.

[12] R. Kumaresan, D. W. Tufts, and L. L. Scharf, "A prony method for noisy data: Choosing the signal components and selecting the order in exponential signal models," *Proc. IEEE*, vol. 72, pp. 230–233, Feb. 1984.

[13] S. V. Kay and A. K. Shaw, "Frequency estimation by principal component AR spectral estimation method without eigen-decomposition," *IEEE Trans. Acoustic Speech Signal Processing*, vol. 36, pp. 95–101, Jan. 1988.

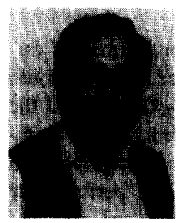
[14] S. J. Orfanidis and L.M. Vail, "Zero-tracking adaptive filters," *IEEE Trans. Acoust., Speech, Signal Processing*, vol. ASSP-34, pp. 1566–1572, Dec. 1986.

[15] J. Stroer and R. Bulirsch, *Introduction to Numerical Analysis.* New York: Springer Verlag, 1980.

[16] G. H. Golub and C. F. Van Loan, *Matrix Computations.* Baltimore: Johns Hopkins University Press, 1990, 2nd ed.

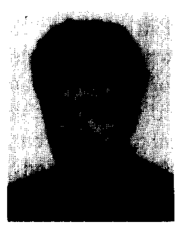
[17] Matlab Reference Guide, *The Mathworks Inc.*, Natick, MA, 1992.

[18] *NAG Fortran Library Routine Document*, ch. C02.



George V. Moustakides (S'79–M'92) received the diploma in electrical engineering from the National Technical University of Athens, Greece, in 1979, the M.Sc. degree in systems engineering from the Moore School of Electrical Engineering, University of Pennsylvania, Philadelphia, in 1980, and the Ph.D. degree in electrical engineering from Princeton University, Princeton, NJ, in 1983.

From 1983 to 1986, he held a research position at the Institut de Resherche en Informatique et Systemes Aleatoires (IRISA-INRIA), Rennes, France and, from 1987 to 1990, a research position at the Computer Technology Institute (CTI) of Patras, Patras, Greece. Since 1991, he has been an Associate Professor in the Department of Computer Engineering and Informatics, University of Patras, Patras, Greece. His interests include detection of signals, detection of changes in systems, adaptive estimation techniques, and biomedical signal processing.



Kostas Berberidis (S'87–M'90) received the Diploma degree in electrical engineering from the University of Thrace, Greece, in 1985 and the Ph.D. degree in computer engineering and informatics from the University of Patras, Greece, in 1990.

From 1986 to 1990, he was a Research Assistant at the Computer Technology Institute (CTI) of Patras, Greece. During 1991, he served in the Greek Army, working in the Speech Processing Laboratory at the National Defense Research Center. Since 1992, he has been with the Computer Technology Institute of Patras, Greece, except for the period from June 1994 to February 1995, where is is currently a Postdoctoral Fellow at the Centre Commun d' Etudes de Telediffusion et Telecommunications (CCETT), Rennes, France. His research interests include adaptive algorithms for digital signal processing, spectral analysis, and communications signal processing.

Dr. Berberidis is a member of the Technical Chamber of Greece.