Design of Two-Dimensional Zero-Phase FIR Filters via the Generalized McClellan Transform

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Abstract—**In this paper we present a design method of two-dimensional (2-D) centrosymmetric zero phase finite impulse response filters, via the generalized McClellan transform. An in depth study of the transform involved in the proposed design method reveals a number of useful properties. These properties are used in the design method for an optimal definition of the generalized McClellan transform coefficients. The method can be applied to the design of classical 2-D FIR filters, yielding filters that are very close to the 2-D ideal specifications.**

I. Introduction

The design of two-dimensional (2-D) digital filters has received growing interest over the last few years. This is due to the variety of applications in fields such as image processing, medical diagnosis, planetary physics, industrial inspection, radar, sonar, seismic, geophysical data processing, etc.

Design approaches for 2-D finite impulse response (FIR) and infinite impulse response (IIR) digital filters can be broadly classified into two categories.

i) Based on transformations of 1-D filters [8]–[19].
ii) Based on direct min-max and L_p optimization techniques [20]–[27].

The design approaches of the IIR filters [1]–[5], [16]–[24] are generally more complicated than the corresponding approaches for FIR filters, since stability constraints must also be considered.

Category i) has the property that the design problem can be divided into two decoupled design subproblems; namely the selection of a high-order 1-D filter and the selection of a low-order 2-D transform, thereby allowing high-order 2-D designs to be obtained with small computational effort. Furthermore, these filters can be often designed to be optimal in the Chebyshev sense and can be efficiently implemented [6], [7].

One very well-known transform is the McClellan transform. The original McClellan transform was introduced in [8] and was proved to be a very useful tool for the design of circular filters with low cut-off radius and also for the 45° 2-D quadrantal fan filter. Because of the simplicity of the McClellan transform, after its introduction in [8], important extensions were made in several directions. In [9], the original McClellan transform was generalized, the necessity of scaling of the transform was introduced, and a design method for filters that possess quadrantal symmetry was presented. The disadvantages of this method were the suboptimal definition of the McClellan transform coefficients and the necessity of redesigning the 1-D prototype filter because of the necessary scaling [10]. Later in [11], a computer-aided design optimization method was proposed for the design of circularly symmetric 2-D filters. In [12] and [13] the problem of scaling was considered and a scaling-free McClellan transform was given that is appropriate for the design of circular and quadrantally elliptical filters. In [14], a fast design method was proposed, which gives an approximate solution to the design of 2-D elliptical filters. Finally in [15] a method for designing quadrantal fan filters was proposed. The coefficients of the McClellan transform involved are obtained by solving a well-defined optimization problem. The resulting transform does not need any scaling.

The goal of this paper is to present a design for 2-D zero-phase FIR digital filters based on the generalized McClellan transform of [14]. Specifically we will generalize the design method that was presented in [15]. The proposed method derives filters that are very close to the ideal specifications, for most classical 2-D zero-phase FIR filters.

The paper is organized as follows. Section II contains a brief presentation of the generalized McClellan transform and certain properties it satisfies that are important for the design procedure. In Section III we present our design method. In Section IV we apply our method to the design of 2-D quadrantal and centrosymmetric zero-phase FIR filters. Finally, Section V contains the conclusion.

II. The Generalized McClellan Transform

Consider a 1-D zero phase FIR symmetric digital filter of odd length that has frequency response $\hat{G}(e^{j\omega})$. We know then that $\hat{G}(e^{j\omega}) = G(\cos(\omega))$. The McClellan transform method uses the relation

$$\cos(\omega) = F_{GM}(\omega_1, \omega_2) \tag{2.1}$$

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where ω is the 1-D frequency, (ω_1, ω_2) is the 2-D frequency pair, and $F_{GM}(\omega_1, \omega_2)$ ("GM" stands for generalized McClellan transform) following [14] is defined as

$$F_{GM}(\omega_1, \omega_2) = \sum_{i=0} \sum_{j=0} t_{ij} \cos(i\omega_1) \cos(j\omega_2) + \sum_{k=1} \sum_{l=1} s_{kl} \sin(k\omega_1) \sin(l\omega_2). \quad (2.2)$$

Notice from (2.1) that a necessary condition that $F_{GM}(\omega_1, \omega_2)$ must satisfy is

$$-1 \leqslant F_{GM}(\omega_1, \omega_2) \leqslant 1 \qquad \forall\, \omega_1, \omega_2 \in [-\pi, \pi]. \quad (2.3)$$

Usually we like in (2.3) to have points (ω_1, ω_2) that satisfy the equalities, in order to cover the whole 1-D frequency band.

In this paper we will concentrate on the generalized McClellan transform of the first order, given by

$$f(\omega_1, \omega_2) = t_{00} + t_{10}\cos(\omega_1) + t_{01}\cos(\omega_2) + t_{11}\cos(\omega_1)\cos(\omega_2) + s_{11}\sin(\omega_1)\sin(\omega_2). \quad (2.4)$$

Substituting in $G(\cos(\omega))$ the $\cos(\omega)$ from (2.4), the 1-D frequency response $G(\cos(\omega))$ is transformed into a 2-D frequency response $\hat{H}(e^{j\omega_1}, e^{j\omega_2})$, which is centrosymmetric. Notice that if the impulse response of the 1-D prototype filter is to extend $(2M+1)$, then the impulse response $h(n,m)$ (which corresponds to the 2-D frequency response $\hat{H}(e^{j\omega_1}, e^{j\omega_2})$) is to extend $(2M+1)\times(2M+1)$.

The basic property used in design methods via transformations is that the isopotentials of the transform (in the 2-D frequency plane) are also isopotentials for the frequency response of the filter. This is clear since an isopotential of the transform corresponds to a constant $\cos(\omega)$ and thus to a constant $G(\cos(\omega))$. Based on this fact we can separate the design procedure into two parts: namely the design of the 1-D prototype filter and the design of the transform. In this paper we will concentrate only on the second part since for the first there already exist very powerful methods. The only quantity needed in designing the 1-D prototype filter (usually low-pass) is the definition of its cut-off frequency. This frequency will be explicitly defined with respect to the optimum transform coefficients.

Let us now concentrate on the transform defined in (2.4). We would like $f(\omega_1, \omega_2)$ to satisfy (2.3), that is,

$$-1 \leqslant f(\omega_1, \omega_2) \leqslant 1 \qquad \forall\, \omega_1, \omega_2 \in [-\pi, \pi]. \quad (2.5)$$

Notice that in order for $f(\omega_1, \omega_2)$ to satisfy (2.5) and also to have full coverage of the 1-D frequency band, it is necessary and sufficient that the maximum and minimum values of the transform equal 1 and -1, respectively. Notice also that if we know the maximum and minimum values of the transform (2.4) (say $f_{\max}$ and $f_{\min}$, respectively), then we can define a scaled transform $F(\omega_1, \omega_2)$, that has maximum and minimum values equal to 1 and -1, as follows:

$$F(\omega_1, \omega_2) = c_1 f(\omega_1, \omega_2) - c_2 \quad (2.6a)$$

where

$$c_1 = \frac{2}{f_{\max} - f_{\min}}$$
$$c_2 = \frac{f_{\max} + f_{\min}}{f_{\max} - f_{\min}}. \quad (2.6b)$$

This new scaled transform is again a generalized McClellan transform of the form (2.4) having exactly the same isopotentials with the original one (but corresponding of course to different constants). To distinguish f from F, we will call f the unscaled and F the scaled transform. From the above we conclude that the determination of the extrema of the unscaled transform (2.4) is vital for proceeding in our design method. With the next theorem we give the maximum and minimum values of $f(\omega_1, \omega_2)$.

Theorem: The maximum and minimum values of the unscaled generalized McClellan transform (2.4) are given by

$$f_{\max}(t_{00}, t) = \begin{cases} t_{00} + \dfrac{1}{s_{11}^2 - t_{11}^2}\Big(t_{10}t_{01}t_{11} \\ \quad + |s_{11}|\sqrt{(t_{10}^2 + s_{11}^2 - t_{11}^2)(t_{01}^2 + s_{11}^2 - t_{11}^2)}\Big), \\ \quad \text{when } s_{11}^2 \geqslant t_{11}^2 \\ + \max\{|t_{10}|, |t_{01}|\}\big(\max\{|t_{11}|, \min\{|t_{10}|, |t_{01}|\}\} \\ \quad + \epsilon \min\{|t_{11}|, |t_{10}|, |t_{01}|\}\big) \\ + \epsilon |t_{11}| \min\{|t_{10}|, |t_{01}|\} \\ t_{00} + |t_{10}| + |t_{01}| + |t_{11}| \\ - (1-\epsilon)\min\{|t_{10}|, |t_{01}|, |t_{11}|\}, \text{ otherwise} \end{cases} \quad (2.7)$$

$$f_{\min}(t_{00}, t) = \begin{cases} t_{00} + \dfrac{1}{s_{11}^2 - t_{11}^2}\Big(t_{10}t_{01}t_{11} \\ \quad - |s_{11}|\sqrt{(t_{10}^2 + s_{11}^2 - t_{11}^2)(t_{01}^2 + s_{11}^2 - t_{11}^2)}\Big), \\ \quad \text{when } s_{11}^2 \geqslant t_{11}^2 \\ + \max\{|t_{10}|, |t_{01}|\}\big(\max\{|t_{11}|, \min\{|t_{10}|, |t_{01}|\}\} \\ \quad - \epsilon \min\{|t_{11}|, |t_{10}|, |t_{01}|\}\big) \\ \quad - \epsilon |t_{11}| \min\{|t_{10}|, |t_{01}|\} \\ t_{00} - |t_{10}| - |t_{01}| - |t_{11}| \\ + (1+\epsilon)\min\{|t_{10}|, |t_{01}|, |t_{11}|\}, \text{ otherwise} \end{cases} \quad (2.8)$$

where

$$t^T = [t_{10} \quad t_{01} \quad t_{11} \quad s_{11}] \quad (2.9a)$$

and

$$\epsilon = \operatorname{sign}(t_{10}t_{01}t_{11}). \quad (2.9b)$$

Proof: For a proof of the theorem see the Appendix.

Let us define

$$\Delta f(t) = f_{\max}(t_{00}, t) - f_{\min}(t_{00}, t) \quad (2.10)$$

the difference that appears in the scaled transform (2.6). Using the results of the above theorem $\Delta f(t)$ can take

TABLE I
THE DIFFERENT FORMS OF $\Delta f(t)$

#	$\Delta f(t)$	Constraints
1	$2(\lvert t_{10}\rvert+\lvert t_{01}\rvert+\lvert t_{11}\rvert-\min\{\lvert t_{11}\rvert,\lvert t_{10}\rvert,\lvert t_{01}\rvert\})$	$\max\{\lvert t_{10}\rvert,\lvert t_{01}\rvert\}(\max\{\lvert t_{11}\rvert,\min\{\lvert t_{10}\rvert,\lvert t_{01}\rvert\}\}$ $-\epsilon\min\{\lvert t_{11}\rvert,\lvert t_{10}\rvert,\lvert t_{01}\rvert\})$ $-\epsilon\lvert t_{11}\rvert\min\{\lvert t_{10}\rvert,\lvert t_{01}\rvert\}+t_{11}^2\geqslant s_{11}^2$
2	$\lvert t_{10}\rvert+\lvert t_{01}\rvert+\lvert t_{11}\rvert+\frac{1}{t_{11}^2-s_{11}^2}\left(t_{10}t_{01}t_{11}-\lvert s_{11}\rvert\sqrt{(t_{10}^2+s_{11}^2-t_{11}^2)(t_{01}^2+s_{11}^2-t_{11}^2)}\right)$	$\max\{\lvert t_{10}\rvert,\lvert t_{01}\rvert\}(\max\{\lvert t_{11}\rvert,\min\{\lvert t_{10}\rvert,\lvert t_{01}\rvert\}\}$ $-\epsilon\min\{\lvert t_{11}\rvert,\lvert t_{10}\rvert,\lvert t_{01}\rvert\})$ $-\epsilon\lvert t_{11}\rvert\min\{\lvert t_{10}\rvert,\lvert t_{01}\rvert\}+t_{11}^2<s_{11}^2<$ $(\lvert t_{11}\rvert+\lvert t_{10}\rvert)(\lvert t_{11}\rvert+\lvert t_{01}\rvert)$
3	$\frac{2\lvert s_{11}\rvert}{s_{11}^2-t_{11}^2}\sqrt{(t_{10}^2+s_{11}^2-t_{11}^2)(t_{01}^2+s_{11}^2-t_{11}^2)}$	$(\lvert t_{11}\rvert+\lvert t_{10}\rvert)(\lvert t_{11}\rvert+\lvert t_{01}\rvert)\leqslant s_{11}^2$

three different forms, which are presented in Table I. Notice that, even though the two extrema are functions of t_{00} and t, $\Delta f(t)$ is a function only of t.

Comments: Let us now make some comments related to the contents of Table I. The first form of $\Delta f(t)$ is valid, when the transform (2.4) attains its extrema on the border of the first or second quadrant. Specifically for the case of quadrantal symmetry, i.e., $s_{11}=0$, as it can be seen from Table I, we can only have this form. The second form corresponds to the case where the transform (2.4) attains one of its extrema on the border of a quadrant and the other in the interior of a quadrant. More precisely, if $\epsilon=1$, then $f_{\max}$ is located on the border of a quadrant and $f_{\min}$ in the interior of a quadrant and when $\epsilon=-1$, $f_{\max}$ and $f_{\min}$ are located in the opposite way. Finally, the third form is valid when the transform attains both extrema in the interior of some quadrant.

Equations (2.7) and (2.8) and Table I will play an important role in defining our design procedure in the next section. As we said, we will concentrate only in the design of the transform, namely the definition of the coefficients of $F(\omega_1,\omega_2)$.

III. Defining the Optimality Criterion

In this section we will give a means for defining the coefficients of the scaled generalized McClellan transform (2.6), in order to meet the design requirements of a 2-D zero-phase FIR filter. As we know, in any 2-D filter specification, there is a curve C that separates the passband from the stopband region. Let us call this curve the cut-off curve. One would like to select the coefficients of the transform (2.6) and the 1-D cut-off frequency ω_0 such that the resulting isopotential corresponding to ω_0 approximates the cut-off curve C. This problem tends to be highly nonlinear. We will follow a somewhat different approach introduced in [15]. This approach will yield an optimization problem that is still nonlinear but we believe much easier to solve. Let us assume that a cut-off curve C is given as a collection of pairs (ω_1,ω_2). Instead of looking for coefficients that yield an isopotential that approximates C, we look for coefficients that yield a transform that has a constant value on C; namely,

$$F(\omega_1,\omega_2)\big|_{(\omega_1,\omega_2)\in C}=\text{constant}. \tag{3.1}$$

Unfortunately, it is not possible to satisfy (3.1) in general. We will thus try to satisfy this requirement in some approximate sense. To this end let us define the mean value of $F(\omega_1,\omega_2)$ on the cut-off curve C by

$$\bar{F}=\frac{1}{L}\oint_C F(\omega_1,\omega_2)\,ds \tag{3.2}$$

and the variance by

$$V_F=\frac{1}{L}\oint_C\left[F(\omega_1,\omega_2)-\bar{F}\right]^2 ds \tag{3.3}$$

where all integrals are line integrals and $L=\oint_C ds$ is the length of the cut-off curve C. Notice that if $V_F=0$ then (3.1) is true. Thus the closer V_F is to zero, the better we approximate (3.1). It is exactly in this sense that we will try to optimize the selection of the coefficients. In other words we will minimize the variance V_F with respect to the coefficients of the scaled McClellan transform (2.6).

Remember that $F(\omega_1,\omega_2)$ is a scaled transform ($F_{\max}=1$, $F_{\min}=-1$) and thus its coefficients are constrained because of these two relations. We would like to express $\bar{F}$ and V_F with respect to the unscaled transform (2.4) coefficients. Substitution of (2.6) in (3.2) and (3.3) gives

$$\bar{F}=\frac{c_1}{L}\oint_C f(\omega_1,\omega_2)\,ds-c_2 \tag{3.4}$$

and

$$V_F=\frac{c_1^2}{L}\oint_C\left[f(\omega_1,\omega_2)-\frac{1}{L}\oint_C f(\omega_1,\omega_2)\,ds\right]^2 ds \tag{3.5}$$

where $f(\cdot)$ and c_1,c_2 are given by (2.4) and (2.6b), respectively. Notice that V_F depends only on c_1 and thus on the difference $\Delta f(t)$ and not on c_2.

Substituting (2.4) into (3.4) yields

$$\bar{F}=c_1(t_{00}+A_{10}t_{10}+A_{01}t_{01}+A_{11}t_{11}+B_{11}s_{11})-c_2 \tag{3.6}$$

where

$$A_{10} = \frac{1}{L}\oint_C \cos(\omega_1)\, ds$$
$$A_{01} = \frac{1}{L}\oint_C \cos(\omega_2)\, ds$$
$$A_{11} = \frac{1}{L}\oint_C \cos(\omega_1)\cos(\omega_2)\, ds$$
$$B_{11} = \frac{1}{L}\oint_C \sin(\omega_1)\sin(\omega_2)\, ds. \tag{3.7}$$

Notice that the A_{ij}, $i, j = 0, 1$ and B_{11} are independent of the transform coefficients and depend only on the cut-off curve C. They can thus be precomputed. Now, using (3.6), (3.5) can be written as follows:

$$V_F = \frac{c_1^2}{L}\oint_C [D_{10}(\omega_1,\omega_2)t_{10} + D_{01}(\omega_1,\omega_2)t_{01} + D_{11}(\omega_1,\omega_2)t_{11} + E_{11}(\omega_1,\omega_2)s_{11}]^2\, ds \tag{3.8}$$

where

$$D_{10}(\omega_1,\omega_2) = \cos(\omega_1) - A_{10}$$
$$D_{01}(\omega_1,\omega_2) = \cos(\omega_2) - A_{01}$$
$$D_{11}(\omega_1,\omega_2) = \cos(\omega_1)\cos(\omega_2) - A_{11}$$
$$E_{11}(\omega_1,\omega_2) = \sin(\omega_1)\sin(\omega_2) - B_{11}. \tag{3.9}$$

Defining the vector

$$D^T(\omega_1,\omega_2) = [D_{10}(\omega_1,\omega_2) \quad D_{01}(\omega_1,\omega_2) \quad D_{11}(\omega_1,\omega_2) \quad E_{11}(\omega_1,\omega_2)] \tag{3.10}$$

and substituting c_1 from (2.6b), V_F can be written as follows:

$$V_F = \frac{4}{L}\frac{t^T Q t}{[\Delta f(t)]^2} \tag{3.11}$$

where t is defined in (2.9a); the denominator $\Delta f(t)$ is given in Table I, and Q is a positive definite matrix defined as

$$Q = \oint_C D(\omega_1,\omega_2) D^T(\omega_1,\omega_2)\, ds. \tag{3.12}$$

Notice that Q is a matrix independent of t and thus can be precomputed for any given cut-off curve C. We can see that the minimization problem is expressed with respect to the coefficients of the unscaled transform (2.4). Also, the coefficient t_{00} does not enter in the definition of V_F because it can only change the mean value $\bar{F}$ and not the variance V_F.

There is a very important property satisfied by V_F; it is a homogeneous function of order zero of the vector t. That is, if we substitute t with λt, then V_F remains unchanged. Based on this property we can prove the following lemma.

Lemma: The minimization problem defined in (3.11) is equivalent to

$$\min_t t^T Q t \tag{3.13}$$

subject to

$$\Delta f(t) = 2. \tag{3.14}$$

If t^* is an optimum vector of this new minimization problem, and we define

$$t_{00}^* = -\frac{f_{\max}(0,t^*) + f_{\min}(0,t^*)}{2} \tag{3.15}$$

then the vector $[t_{00}^* \quad t^*]$ constitutes a set of optimum scaled transform coefficients.

Proof: The proof of the lemma is given in the Appendix.

The minimization problem defined by the lemma is nonlinear basically because of the nonlinear forms of the constraint (3.14) (see Table I). Even though $\Delta f(t)$ can take three different forms, in many design cases it is possible to know *a priori* the regions where the maximum and minimum occur and thus confine ourselves to some of the possible forms. Typical examples are the quadrantal case where the extrema occur on the borders, and thus $\Delta f(t)$ is defined only by the first form of Table I and also the low-pass case where the maximum occurs at the origin, and thus $\Delta f(t)$ can take the first and second forms of Table I.

In order to complete our proposed method we must define the 1-D cut-off frequency ω_0 of the prototype zero-phase FIR filter. Since we like $\cos(\omega_0)$ to correspond to the cut-off curve C, we would like to have

$$F(\omega_1,\omega_2)|_{(\omega_1,\omega_2)\in C} = \cos(\omega_0). \tag{3.16}$$

Because this is not possible, the best we can do is to equate $\cos(\omega_0)$ to the mean value $\bar{F}$, that is,

$$\omega_0 = \arccos(\bar{F}) \tag{3.17}$$

where from (3.6) the mean value $\bar{F}$ is given by

$$\bar{F} = t_{00}^* + A_{10}t_{10}^* + A_{01}t_{01}^* + A_{11}t_{11}^* + B_{11}s_{11}^* \tag{3.18}$$

and A_{ij}, $i, j = 0, 1$ and B_{11} are defined in (3.7). Equation (3.17) has always a solution because $\bar{F}$ as the mean of a scaled transform is always absolutely bounded by unity.

In the next section we will summarize the basic steps of our design method and we will give some examples of classical 2-D zero-phase FIR filters.

IV. Design Procedure—Applications

In this section we will apply our method of Section III to the design of some common 2-D zero-phase FIR filters. Before going to the applications let us summarize the steps we must follow in order to design a 2-D zero-phase FIR filter with our method.

Step 1: Using (3.7), (3.9), and (3.12), we compute the elements of the matrix Q. By minimizing (3.13) subject to (3.14) and using (3.15) we obtain the optimum coefficient vector $[t_{00}^* \quad t^*]$ which is associated with the scaled McClellan transform.

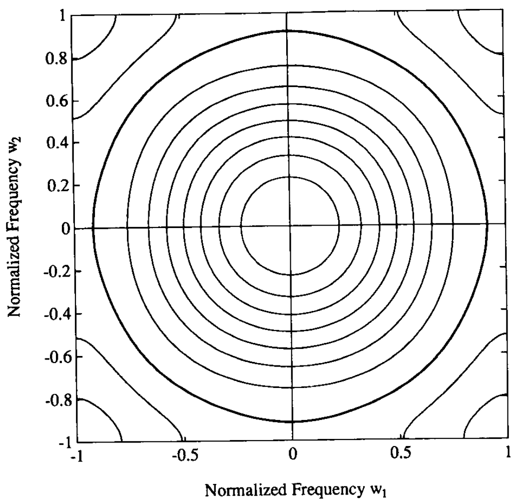

Fig. 1. Isopotentials of a circular filter with cut-off radius $10\pi/11$.

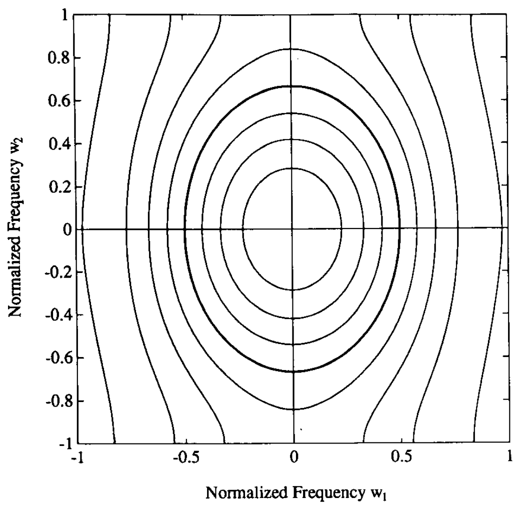

Fig. 2. Isopotentials of a quadrantally elliptic filter with axes $\alpha = \pi/2$, $\beta = 2\pi/3$.

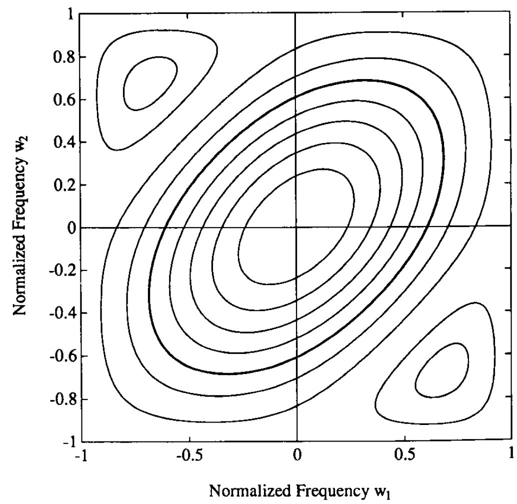

Fig. 3. Isopotentials of a centrosymmetric elliptic filter with axes $\alpha = 5\pi/6$, $\beta = \pi/2$, and $\theta = 45°$.

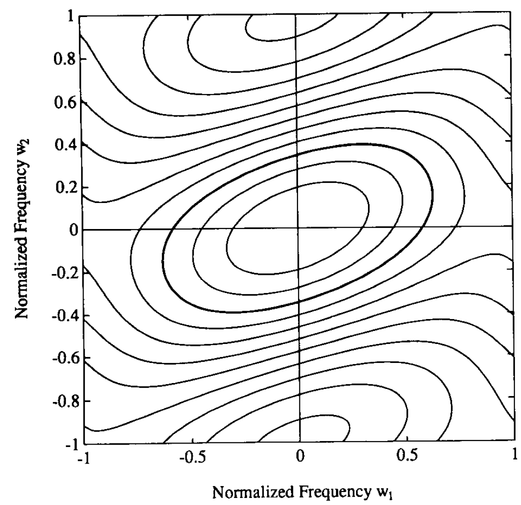

Fig. 4. Isopotentials of a centrosymmetric elliptic filter with axes $\alpha = 2\pi/3$, $\beta = \pi/3$, and $\theta = 20°$.

Step 2: From (3.17) we compute the desired cut-off frequency ω_0 of the 1-D prototype filter. Using some 1-D filter design method, we design a zero-phase FIR filter with cut-off frequency ω_0.

Step 3: We express the frequency response of the 1-D designed filter in the form $\hat{G}(e^{j\omega}) = \Sigma_n q_n T_n[\cos(\omega)]$ where $T_n[x]$ is the nth-order Chebyshev polynomial. We then replace the $\cos(\omega)$ in the frequency response with the generalized McClellan transform (2.6) to produce the desired 2-D zero-phase FIR filter.

We will now apply our method to the design of seven filters, most of which are classical and widely used in 2-D signal processing applications. In order to measure the quality of our design, we will use as measure of performance the relative error E in the areas of the designed and ideal passbands, that is,

$$E = \frac{\iint_{(R_D \cup R_I)-(R_D \cap R_I)} d\omega_1\, d\omega_2}{\iint_{R_I} d\omega_1\, d\omega_2} \times 100 \qquad (4.1)$$

where R_D and R_I are the designed and the ideal passband regions, respectively.

We have classified the designed filters into circular, quadrantally elliptical, centrosymmetric elliptical, centrosymmetric fan, and quadrant filters. In Figs. 1–7 we have plotted the resulting isopotentials. Table II contains

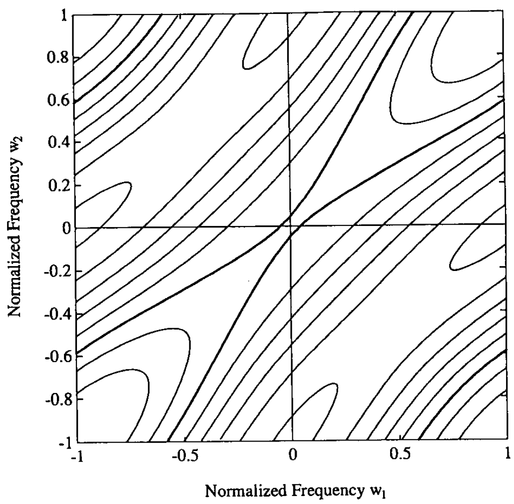

Fig. 5. Isopotentials of a centrosymmetric fan filter with $\theta_1 = 30°$ and $\theta_2 = 60°$.

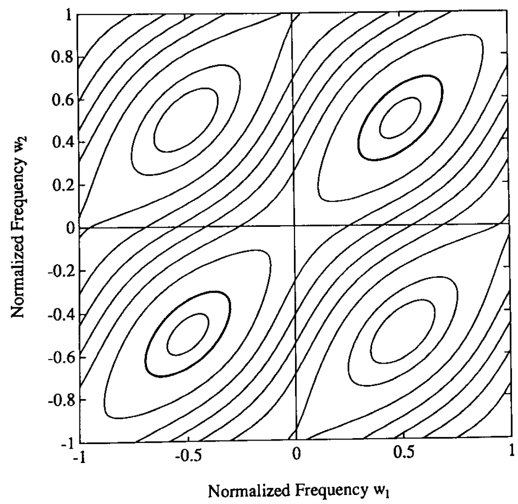

Fig. 7. Isopotentials of a quadrant elliptical filter with $\alpha = \pi/4$, $\beta = \pi/8$, $\theta = 45°$, centered at $(\pi/2, \pi/2)$.

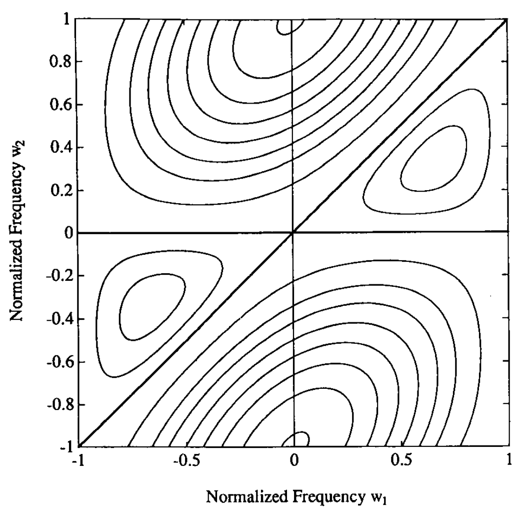

Fig. 6. Isopotentials of a centrosymmetric fan filter with $\theta_1 = 0°$ and $\theta_2 = 45°$.

the values of the scaled transform coefficients, the corresponding 1-D cut-off frequency ω_0 and the relative error E for each of the designed filters. Let us now comment on each class separately.

i) Circular 2-D FIR Filters: In order to point out the possibilities of our method for this class we have selected a somehow worst case situation, that is, the case of a large cut-off radius. Specifically, our choice was a cut-off radius of $10\pi/11$. From Table II we can see that the designed filter has a very small relative error E. In Fig. 1 we have plotted the resulting isopotentials. Notice that the cut-off curve (indicated by the fat line in Fig. 1) is very close to a circle. For the class of circular filters our method has an excellent performance.

ii) Quadrantally Elliptical 2-D FIR Filters: This is another useful class of 2-D FIR filters. For this case our method yields again excellent results with small relative errors E. As an example we have designed a filter with minor axis $\alpha = \pi/2$ and major axis $\beta = 2\pi/3$. Fig. 2 has the resulting isopotentials.

iii) Centrosymmetric Elliptical 2-D FIR Filters: Such filters are needed in many practical situations of 2-D signal processing [14]. The performance of our method for this class depends heavily on the magnitude of the major axis, the ratio of the major to minor axes, and the orientation of the ellipse. More specifically, for a given orientation and major axis magnitude, there exists an upper limit for the ratio for which the relative error E is acceptable. This upper limit is a decreasing function of the magnitude of the major axis. For the symmetric cases, i.e., orientation angle $\theta = 0°, 45°, 90°$, there is practically no upper limit for the ratio. If the angle θ though takes an arbitrary value, the upper limit exists and restricts the design of any type of ellipse. In any case, if the major axis has values of the order of $\pi/2$ we can practically design any ellipse. We give two examples for this class of filters. The first example has major axis $\alpha = 5\pi/6$, minor axis $\beta = \pi/2$, and $\theta = 45°$ (symmetric orientation). From Fig. 3 and Table II we can see that the performance is very good. The second example has $\alpha = 2\pi/3$, $\beta = \pi/3$, and $\theta = 20°$ (nonsymmetric orientation). From Fig. 4 and Table II we can see that the performance is acceptable.

iv) Centrosymmetric 2-D Fan Filters: This is a class of 2-D filters that is useful for seismic and geophysical data processing applications [19]–[22]. A fan filter is completely defined if we define the orientation angles θ_1, θ_2 of its passband. For this class we will present two filters. The first has a passband width of 30° with $\theta_1 = 30°$, $\theta_2 = 60°$, symmetrically oriented with respect to the main diagonal of the first quadrant. In Fig. 5 we have plotted

TABLE II
OPTIMUM SCALED PARAMETERS AND RELATIVE ERROR OF THE IDEAL AND DESIGNED PASSBANDS

#	t_{00}	t_{10}	t_{01}	t_{11}	s_{11}	ω_0	$E(\%)$
1	−0.3955	0.5	0.5	0.3955	0	2.4325	0.49
2	−0.3124	0.6640	0.3360	0.3124	0	1.5456	0.22
3	−0.3420	0.4542	0.4542	0.4336	−0.4150	1.7546	1.05
4	−0.0720	0.0720	0.6431	0.3569	−0.2760	1.3697	2.55
5	0.1820	0.1822	0.1820	−0.8178	0.7031	1.8392	1.50
6	0.3346	−0.4449	0.4449	0.4449	−0.4449	0.6769	0
7	0	0	0	0.5765	−1.0000	0.4961	1.15

the isopotentials for this case. The resulting cut-off curve approximates the ideal very closely in the first and third quadrants. Notice though that there exists a small part of the designed cut-off curve (and thus of the passband) that appears in the second and fourth quadrants. These unwanted parts can be eliminated by cascading a quadrant fan filter [19]. The relative error E that appears in Table II does not take into account these unwanted parts. The second filter has a passband width of 45° with $\theta_1 = 0°$, $\theta_2 = 45°$. Here our method has zero error, which means that our solution is exact. In Fig. 6 we can see the resulting isopotentials. Notice that for the second filter we do not have the problem that occurred in the first example.

v) Quadrant 2-D FIR Filters: This class constitutes a generalization of the quadrant fan filter. We present one example for this class just to demonstrate the capabilities of our method and not because we know some specific application of this filter. We will design a quadrant elliptical filter with axes $\alpha = \pi/4$, $\beta = \pi/8$, $\theta = 45°$ and with its center located at $(\pi/2, \pi/2)$. The resulting isopotentials are shown in Fig. 7.

Comments: Our proposed method can be applied to any design specifications and will yield the best possible McClellan transform coefficients. Notice that it also defines in some optimum way the required 1-D cut-off frequency ω_0. This last characteristic is not present in other existing methods [9], [12]–[14]. Comparing our method to [12]–[14] for the design of elliptic filters, the results are comparable for most design cases. There is, though, an important difference when we design large ellipses as in the case of our first example. For these cases our method gives better results basically because of the optimum definition of the 1-D cut-off frequency ω_0, which in [12]–[14] is arbitrarily equated to the length of the minor axis of the ellipse.

V. CONCLUSION

In this paper a new design method for 2-D FIR filters was presented. It uses the generalized McClellan transform to map the 1-D frequency axis to the 2-D frequency plane. The coefficients of the scaled generalized McClellan transform are selected by solving a well-defined optimization problem. This problem takes into account the necessary scaling needed in order to cover the whole 1-D frequency axis. The resulting optimization problem is nonlinear but it has only four parameters. The proposed method applied to conventional filters such as cyclic, elliptic, and fan, gives excellent results.

APPENDIX A

Proof of the Theorem

Our goal is to find the global maximum and global minimum values of the generalized McClellan transform (2.4). To this end let us consider the following two cases.

Consider first the case $|t_{10}| \geq |t_{01}|$. Using the identity $\alpha\cos(x) + \beta\sin(x) = \cos(x+\phi)\sqrt{\alpha^2+\beta^2}$ we can write $f(\omega_1, \omega_2)$ as follows:

$$f(\omega_1,\omega_2) = t_{00} + t_{10}\cos(\omega_1) + \cos(\omega_2 - \phi(\omega_1)) \cdot \sqrt{(t_{11}^2 - s_{11}^2)\cos^2(\omega_1) + 2t_{01}t_{11}\cos(\omega_1) + t_{01}^2 + s_{11}^2} \quad (A.1)$$

where $\phi(\omega_1)$ is defined as the common solution of

$$\sin(\phi(\omega_1)) = \frac{s_{11}\sin(\omega_1)}{\sqrt{(t_{11}^2 - s_{11}^2)\cos^2(\omega_1) + 2t_{01}t_{11}\cos(\omega_1) + t_{01}^2 + s_{11}^2}}$$

$$\cos(\phi(\omega_1)) = \frac{t_{01} + t_{11}\cos(\omega_1)}{\sqrt{(t_{11}^2 - s_{11}^2)\cos^2(\omega_1) + 2t_{01}t_{11}\cos(\omega_1) + t_{01}^2 + s_{11}^2}}. \quad (A.2)$$

Let us fix the value of ω_1 in (A.1) and find the extrema of $f(\omega_1, \omega_2)$ with respect to ω_2. We can easily see that we have maximum for $\omega_2 = \phi(\omega_1)$ and minimum for $\omega_2 = \pi - \phi(\omega_1)$. Let us concentrate only on the computation of the maximum since in a similar way we can compute the minimum. We thus have that

$$f(\omega_1,\omega_2) \leq t_{00} + t_{10}\cos(\omega_1) + \sqrt{(t_{11}^2 - s_{11}^2)\cos^2(\omega_1) + 2t_{01}t_{11}\cos(\omega_1) + t_{01}^2 + s_{11}^2} = \hat{f}(\omega_1). \quad (A.3)$$

As we said, the right-hand side value is achieved by selecting $\omega_2 = \phi(\omega_1)$. The function $\hat{f}(\omega_1)$ is a function only of ω_1 and we will try to find its maximum. Let us define $\Omega_1 = \cos(\omega_1)$ then (A.3) can be written as follows:

$$\hat{f}(\Omega_1) = t_{00} + t_{10}\Omega_1 + \sqrt{(t_{11}^2 - s_{11}^2)\Omega_1^2 + 2t_{01}t_{11}\Omega_1 + t_{01}^2 + s_{11}^2} \quad \text{(A.4)}$$

where $-1 \leqslant \Omega_1 \leqslant 1$. Computing the first and the second derivatives of (A.4), we obtain

$$\hat{f}'(\Omega_1) = t_{10} + \frac{(t_{11}^2 - s_{11}^2)\Omega_1 + t_{01}t_{11}}{\sqrt{(t_{11}^2 - s_{11}^2)\Omega_1^2 + 2t_{01}t_{11}\Omega_1 + t_{01}^2 + s_{11}^2}} \quad \text{(A.5}\alpha\text{)}$$

and

$$\hat{f}''(\Omega_1) = \frac{-s_{11}^2(s_{11}^2 - t_{11}^2 + t_{01}^2)}{\left((t_{11}^2 - s_{11}^2)\Omega_1^2 + 2t_{01}t_{11}\Omega_1 + t_{01}^2 + s_{11}^2\right)^{3/2}}. \quad \text{(A.5}\beta\text{)}$$

Since the denominator of (A.5β) is positive for every value of Ω_1 in the interval $[-1,1]$, we conclude that the sign of the $\hat{f}''(\Omega_1)$ is constant and equal to the sign of the numerator. In other words the function $\hat{f}(\Omega_1)$ is either a concave or a convex function of Ω_1 depending on the sign of the term $-s_{11}^2(s_{11}^2 - t_{11}^2 + t_{01}^2)$. Because of the concavity or convexity of $\hat{f}(\Omega_1)$, the function attains its maximum either at the extremities of Ω_1, that is $\Omega_1 = \pm 1$ or $\hat{f}(\Omega_1)$ has a unique maximum inside the interval $(-1,1)$. This last case occurs if and only if the following inequalities hold simultaneously:

$$\hat{f}'(-1) \geqslant 0 \geqslant \hat{f}'(1). \quad \text{(A.6)}$$

Let us now express these inequalities in terms of the coefficients of the transform $f(\omega_1, \omega_2)$. Considering the cases $|t_{11}| \geqslant |t_{01}|$ and $|t_{11}| < |t_{01}|$, we easily obtain that (A.6) holds if and only if

$$s_{11}^2 \geqslant t_{11}^2 + |t_{10}|\left(\max\{|t_{11}|,|t_{01}|\} + \epsilon \min\{|t_{11}|,|t_{01}|\}\right) + \epsilon|t_{11}||t_{01}| \quad \text{(A.7)}$$

where

$$\epsilon = \operatorname{sign}(t_{10}t_{01}t_{11}). \quad \text{(A.8)}$$

We are now in a position to compute the maximum.

If inequality (A.7) holds, then the maximum is achieved for Ω_1^* satisfying

$$\hat{f}'(\Omega_1^*) = 0. \quad \text{(A.9)}$$

The solution of this equation gives

$$\Omega_1^* = \frac{1}{s_{11}^2 - t_{11}^2}\left(t_{01}t_{11} + |s_{11}|t_{10}\sqrt{\frac{t_{01}^2 + s_{11}^2 - t_{11}^2}{t_{10}^2 + s_{11}^2 - t_{11}^2}}\right) \quad \text{(A.10)}$$

with corresponding maximum value

$$\hat{f}_{\max} = t_{00} + \frac{1}{s_{11}^2 - t_{11}^2}\left(t_{10}t_{01}t_{11} + |s_{11}|\sqrt{(t_{10}^2 + s_{11}^2 - t_{11}^2)(t_{01}^2 + s_{11}^2 - t_{11}^2)}\right). \quad \text{(A.11)}$$

If inequality (A.7) does not hold, the maximum is one of the values $\hat{f}(-1), \hat{f}(1)$. Equation (A.4) for $\Omega_1 = \pm 1$ yields

$$\hat{f}(1) = t_{00} + t_{10} + |t_{11} + t_{01}|$$
$$\hat{f}(-1) = t_{00} - t_{10} + |t_{11} - t_{01}|. \quad \text{(A.12)}$$

By considering again the cases $|t_{11}| \geqslant |t_{01}|$ and $|t_{11}| < |t_{01}|$ and using the identity $x = |x|(1 + \operatorname{sign}(x)) - |x|$, we obtain

$$\hat{f}_{\max} = t_{00} + |t_{10}| + |t_{01}| + |t_{11}| - (1 - \epsilon)\min\{|t_{11}|,|t_{01}|\}. \quad \text{(A.13)}$$

Considering now the case $|t_{10}| \leqslant |t_{01}|$ and expressing $f(\omega_1, \omega_2)$ as

$$f(\omega_1, \omega_2) = t_{00} + t_{01}\cos(\omega_2) + \cos(\omega_1 - \phi(\omega_2)) \cdot \sqrt{(t_{11}^2 - s_{11}^2)\cos^2(\omega_2) + 2t_{10}t_{11}\cos(\omega_2) + t_{10}^2 + s_{11}^2} \quad \text{(A.14)}$$

and following a similar approach as before we obtain that when

$$s_{11}^2 \geqslant t_{11}^2 + |t_{01}|\left(\max\{|t_{11}|,|t_{10}|\} + \epsilon \min\{|t_{11}|,|t_{10}|\}\right) + \epsilon|t_{11}||t_{10}| \quad \text{(A.15)}$$

the maximum is achieved for

$$\Omega_2^* = \frac{1}{s_{11}^2 - t_{11}^2}\left(t_{10}t_{11} + |s_{11}|t_{01}\sqrt{\frac{t_{10}^2 + s_{11}^2 - t_{11}^2}{t_{01}^2 + s_{11}^2 - t_{11}^2}}\right) \quad \text{(A.16)}$$

with maximum value given again by (A.11).

When inequality (A.15) does not hold then the maximum value of the transform is given by

$$\hat{f}_{\max} = t_{00} + |t_{10}| + |t_{01}| + |t_{11}| - (1 - \epsilon)\min\{|t_{11}|,|t_{10}|\}. \quad \text{(A.17)}$$

Combining inequalities (A.7) and (A.15) and equations (A.13) and (A.17), we obtain (2.7).

Proof of the Lemma: Our goal is to prove that the unconstrained minimization problem (3.11) is equivalent to the constrained problem defined by the lemma. In order to prove the lemma it is enough to show that if t_u minimizes the objective function $V_F(t)$, then there exists a vector t_c satisfying the constraint (3.14) that also minimizes the same function. Thus, if

$$V_F(t_u) \leqslant V_F(t), \quad \forall\, t \neq 0$$

define $t_c = \lambda_c t_u$ with $\lambda_c = 2/\Delta f(t_u)$. From the homogeneity of $V_F(t)$ we have

$$V_F(t_c) = V_F(t_u) \leqslant V_F(t), \quad \forall\, t \neq 0.$$

Since $\Delta f(t)$ is a homogeneous function, of order one of the vector t (see Table I), we have

$$\Delta f(t_c) = \Delta f(\lambda_c t_u) = \lambda_c \Delta f(t_u) = 2$$

and thus we conclude that t_c satisfies the constraint (3.14). This means that in order to minimize $V_F(t)$, it is enough to look among vectors that satisfy $\Delta f(t) = 2$. But for these vectors, minimizing $V_F(t)$ is equivalent to minimizing t^TQt.

If we call t^* a vector solving the constrained minimization of the lemma, notice from (2.6b) that it will satisfy $c_1 = 1$. In order now to obtain an optimum scaled transform we must also have $c_2 = 0$ or equivalently

$$f_{\max}(t_{00}^*, t^*) + f_{\min}(t_{00}^*, t^*) = 0. \tag{A.18}$$

Substitution of (2.7) and (2.8) in (A.18) yields the equation that defines t_{00}^*

$$t_{00}^* = -\frac{f_{\max}(0, t^*) + f_{\min}(0, t^*)}{2}. \tag{A.19}$$

This concludes the proof of the lemma.

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