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Design of N-Dimensional Hyperquadrantally Symmetric FIR Filters Using the McClellan Transform

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Abstract—**A method for designing hyperquadrantally symmetric N-D FIR filters is presented. Using the first order McClellan transform to map the N-D frequency space onto the 1-D frequency line it is possible to design hyperquadrantally symmetric N-D filters very accurately. The coefficients of the McClellan transform are selected by optimizing a well defined optimization criterion. Although the proposed method is a generalization to an existing method for the 2-D case, a very elegant and easily computable solution to the optimization problem is presented for the first time.**

I. Introduction and Background Material

In many application areas the need of processing multidimensional signals has been growing considerably over the last years. Two-dimensional and multidimensional techniques have been developed to meet this need. The design of multidimensional filters with prescribed frequency response characteristics is one such technique mainly used to pre-process multidimensional signals.

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A very important class of multidimensional filters is the FIR class. Several methods exist in the literature for the design of this type of filters. We will limit ourselves to design methods that are based on transforms. That is, methods that use transforms to map 1-D filters to multidimensional ones [1]–[12]. Among the most well known transforms is the McClellan transform which has been primarily used for the design of 2-D filters [1], [3], [5], [7]–[12].

In this paper, we present a design method for N-D hyperquadrantally symmetric FIR filters that is based on transforming 1-D filters using the N-D first order symmetric McClellan transform. Our method will follow the design idea of [11] and extend it to the multidimensional case. This extension will be accompanied by a very simple and elegant solution (not having a counterpart in [11]) of the design optimization problem. The proposed method can be applied with success to most important filter design problems.

Let us briefly define the N-D first order symmetric McClellan transform. Let $\mathcal{A}_N = \{0,1\}^N$, denote the N times Cartesian product of the set $\{0,1\}$. This set is composed of 2^N different N-tuples of the form $[i_1 i_2 \cdots i_N]$ where all i_k can take the values 0 or 1. The transform can now be defined as

$$F_N(\omega_1, \omega_2, \cdots, \omega_N) = \sum_{[i_1 i_2 \cdots i_N] \in \mathcal{A}_N} t_{i_1 i_2 \cdots i_N} \cos(i_1\omega_1)\cos(i_2\omega_2)\cdots\cos(i_N\omega_N). \quad (1)$$

Any symmetric 1-D FIR filter can be written in the frequency domain as $H(\cos(\omega))$, where $H(.)$ is a polynomial. The N-D to 1-D mapping (and thus the generation of N-D filters from the 1-D prototype) can be achieved by requiring

$$\cos(\omega) = F_N(\omega_1, \omega_2, \cdots, \omega_N) \quad (2)$$

where ω is the 1-D frequency and $[\omega_1 \cdots \omega_N]$ is a point in the N-D frequency hypercube. To be possible to apply (2) the transform is required to satisfy

$$-1 \le F_N(\omega_1, \omega_2, \cdots, \omega_N) \le 1 \qquad \forall \omega_1, \omega_2, \cdots, \omega_N \in [-\pi, \pi]. \quad (3)$$

In order to use the entire 1-D frequency band the transform must actually attain the two bounds. Such a transform is known as *scaled*. Relation (3) constitutes the main constraint on the transform coefficients.

II. Main Results

In the design methods via transforms the aim is to map the N-D frequency space onto the 1-D frequency line. In order to have a correct design, the N-D passband and stopband regions must correspond through the transform to the 1-D passband and stopband, respectively. In this paper we will be concerned only with N-D design problems that can be mapped to a 1-D lowpass filter. In other words the N-D space must be composed of a single passband and a single stopband region. Notice that for a complete design we must specify the transform coefficients and the 1-D filter. Regarding this last problem (1-D filter) we will be concerned only in specifying the passband and stopband cutoff frequencies ω_p, ω_s. It is then easy to use any known method to find the coefficients of the 1-D filter.

Let us denote by C_p, C_s the $(N-1)$-dimensional passband and stopband cutoff manifolds that separate the regions of interest. We would like to map these two manifolds, with the use of the transform, onto the corresponding 1-D passband and stopband cutoff frequencies ω_p, ω_s. This means

$$F_N(\omega_1,\cdots,\omega_N)|_{[\omega_1\cdots\omega_N]\in C_p} = \cos(\omega_p),$$
$$F_N(\omega_1,\cdots,\omega_N)|_{[\omega_1\cdots\omega_N]\in C_s} = \cos(\omega_s). \quad (4)$$

Clearly the above equalities cannot hold in general. We thus, following an approach similar to [11], require instead the values of the transform to fluctuate, for each manifold, as little as possible around a constant value. In order to make this statement more formal let us define for a manifold C the mean and variance of the transform when $[\omega_1\cdots\omega_N] \in C$

$$\bar{F}_C = \frac{1}{S_C}\int_C F_N(\omega_1,\cdots,\omega_N)\,ds \quad (5)$$

$$V_C = \frac{1}{S_C}\int_C [F_N(\omega_1,\cdots,\omega_N) - \bar{F}_C]^2\,ds. \quad (6)$$

All integrals are surface integrals and ds denotes the surface differential while $S_C = \int_C ds$ the surface "area." Notice that using surface integrals results in mean and variance values that are parametrization invariant. Clearly the role of the constant will play the mean $\bar{F}_C$ and the measure of fluctuation will be the variance V_C. Both quantities are functions of the coefficients of the transform and next we will make this dependence explicit.

Optimization Criterion and Optimal Solution: Expressing the mean and the variance of (6) in terms of the coefficients yields the following:

$$\bar{F}_C = t_{0\cdots 0} + \boldsymbol{t}^T\boldsymbol{a}_C$$
$$V_C(\boldsymbol{t}) = \boldsymbol{t}^T Q_C \boldsymbol{t} \quad (7)$$

where

$$\boldsymbol{t}^T = [t_{100\cdots 0}t_{010\cdots 0}\cdots t_{111\cdots 1}] \quad (8)$$

$$\boldsymbol{a}_C^T = [a_{100\cdots 0}a_{010\cdots 0}\cdots a_{111\cdots 1}] \quad (9)$$

$$a_{i_1\cdots i_N} = \frac{1}{S_C}\int_C \cos(i_1\omega_1)\cdots\cos(i_N\omega_N)\,ds \quad (10)$$

$$Q_C = \frac{1}{S_C}\int_C \boldsymbol{d}(\omega_1,\cdots,\omega_N)\boldsymbol{d}^T(\omega_1,\cdots,\omega_N)\,ds \quad (11)$$

$$\boldsymbol{d}^T(\omega_1,\cdots,\omega_N) = [d_{100\cdots 0}(\omega_1,\cdots,\omega_N)d_{010\cdots 0}(\omega_1,\cdots,\omega_N)\cdots \cdot d_{111\cdots 1}(\omega_1,\cdots,\omega_N)] \quad (12)$$

$$d_{i_1\cdots i_N}(\omega_1,\cdots,\omega_N) = \cos(i_1\omega_1)\cdots\cos(i_N\omega_N) - a_{i_1\cdots i_N}. \quad (13)$$

Notice from (7) that the variance does not depend on the constant coefficient $t_{0\cdots 0}$ of the transform. Also, the matrix Q_C depends only on the selected manifold C and not on the coefficients of the transform and thus for given C it can be considered known.

Since we have two cutoff manifolds, one for the passband C_p and one for the stopband C_s, we can define for each case a different mean and variance. Our intention is, the final transform to have as small as possible variances for both manifolds. Thus we can define as our optimization criterion the following combination

$$V(\boldsymbol{t}) = \boldsymbol{t}^T Q\boldsymbol{t}, \text{with } Q = \alpha_p Q_{C_p} + \alpha_s Q_{C_s} \quad (14)$$

where α_p, α_s are nonnegative weights selected according to which manifold we like to approximate better. After the above definitions it is clear that the problem we like to solve is

$$\min_{\boldsymbol{t}} \boldsymbol{t}^T Q\boldsymbol{t} \quad (15)$$

subject to the constraints

$$\max_{\omega_1,\cdots,\omega_N} F_N(\omega_1,\cdots,\omega_N) = 1, \min_{\omega_1,\cdots,\omega_N} F_N(\omega_1,\cdots,\omega_N) = -1. \quad (16)$$

With the next lemma we relate the constraints in (16) to the transform coefficients.

Lemma: Any first order symmetric McClellan transform attains its extrema at the corners of the positive hypercube. That is for $\omega_i \in \{\pi, 0\}$.

Proof: The lemma can be easily proved using the fact that the transform is a linear function of $\cos(\omega_i)$, we thus omit any further details.

Using the lemma we limit ourselves in searching for the extrema on the 2^N corners of the positive frequency hypercube. According to the lemma each such corner is a possible candidate for an extremum. Evaluating the transform at the kth corner yields

$$F_N(\omega_{1,k},\cdots,\omega_{N,k}) = t_{0\cdots 0} + \boldsymbol{t}^T\boldsymbol{s}_k, \omega_{i,k} \in \{0,\pi\} \quad (17)$$

where $\boldsymbol{s}_k$ is a vector of length $2^N - 1$ with elements taking values 1 or -1. We will call these vectors *corner vectors*. Notice that there exist 2^N of them and that they depend only on the dimension N and not on the transform coefficients. Thus they can be assumed known. The constraints now in (16) can take the form

$$t_{0\cdots 0} + \max_k\{\boldsymbol{s}_k^T\boldsymbol{t}\} = 1, t_{0\cdots 0} + \min_k\{\boldsymbol{s}_k^T\boldsymbol{t}\} = -1. \quad (18)$$

It is clear that if we knew before hand the corners where the extrema occur then optimizing our criterion (the variance $V(\boldsymbol{t})$) would be easy since it involves optimization of a quadratic function under linear equality constraints. With the next theorem we show that by taking all possible corner combinations, solving the corresponding quadratic problems and finally selecting the one with the smallest variance is the solution we are looking for.

Theorem: The solution to the constraint optimization problem defined by (15) and (18) is given by:

Case of Singular Q: For this case

$$\boldsymbol{t}^o = \frac{2\boldsymbol{u}_0}{(\boldsymbol{s}_i^o - \boldsymbol{s}_j^o)^T\boldsymbol{u}_0}, t_{0\cdots 0}^o = -\frac{(\boldsymbol{s}_i^o + \boldsymbol{s}_j^o)^T\boldsymbol{u}_0}{(\boldsymbol{s}_i^o - \boldsymbol{s}_j^o)^T\boldsymbol{u}_0} \quad (19)$$

where $\boldsymbol{u}_0$ is an eigenvector corresponding to the zero eigenvalue of Q and $\boldsymbol{s}_i^o$, $\boldsymbol{s}_j^o$ are corner vectors maximizing the quantity $|(\boldsymbol{s}_i - \boldsymbol{s}_j)^T\boldsymbol{u}_0|$ over i, j.

Case of Nonsingular Q: For this case

$$\boldsymbol{t}^o = \frac{2Q^{-1}(\boldsymbol{s}_i^o - \boldsymbol{s}_j^o)}{(\boldsymbol{s}_i^o - \boldsymbol{s}_j^o)^T Q^{-1}(\boldsymbol{s}_i^o - \boldsymbol{s}_j^o)},$$
$$t_{0\cdots 0}^o = -\frac{(\boldsymbol{s}_i^o + \boldsymbol{s}_j^o)^T Q^{-1}(\boldsymbol{s}_i^o - \boldsymbol{s}_j^o)}{(\boldsymbol{s}_i^o - \boldsymbol{s}_j^o)^T Q^{-1}(\boldsymbol{s}_i^o - \boldsymbol{s}_j^o)} \quad (20)$$

where $\boldsymbol{s}_i^o$, $\boldsymbol{s}_j^o$ are corner vectors that maximize the expression $(\boldsymbol{s}_i - \boldsymbol{s}_j)^T Q^{-1}(\boldsymbol{s}_i - \boldsymbol{s}_j)$ over i, j.

Proof: We will show only the second case. From the two constraints in (18) we conclude that $\max_{i,j}\{|(\boldsymbol{s}_i - \boldsymbol{s}_j)^T\boldsymbol{t}|\} = 2$. Using the Schwarz inequality we have

$$|(\boldsymbol{s}_i - \boldsymbol{s}_j)^T\boldsymbol{t}|^2 \le (\boldsymbol{t}^T Q\boldsymbol{t})\Big((\boldsymbol{s}_i - \boldsymbol{s}_j)^T Q^{-1}(\boldsymbol{s}_i - \boldsymbol{s}_j)\Big) \quad (21)$$

taking maximum over i, j in both sides, yields

$$\boldsymbol{t}^T Q\boldsymbol{t} \ge \frac{4}{\max_{i,j}(\boldsymbol{s}_i - \boldsymbol{s}_j)^T Q^{-1}(\boldsymbol{s}_i - \boldsymbol{s}_j)}. \quad (22)$$

This lower bound is achieved by $\boldsymbol{t}^o$ as one can verify by direct substitution. Finally selecting $t_{0\cdots 0}$ for the transform to assume a value equal to 1 at corner $\boldsymbol{s}_i^o$ (and thus -1 at $\boldsymbol{s}_j^o$), yields the corresponding expression for $t_{0\cdots 0}^o$.

To complete the proof we need to show that at any other corner the value of the obtained transform is smaller, in absolute value, than unity. Let $\boldsymbol{s}_k$ be any corner vector, then we will show that the value of the transform is smaller than 1 (similarly it can be shown that it is larger than -1). Notice that the value of the transform at the corresponding corner, using (20), can be written as $-1+(\boldsymbol{s}_k-\boldsymbol{s}_j^o)^T\boldsymbol{t}^o$. It is thus enough to show that

$$(\boldsymbol{s}_k-\boldsymbol{s}_j^o)^T\boldsymbol{t}^o \leq 2. \tag{23}$$

This can be easily shown to be valid using Schwarz inequality and the definition of the optimal corner vectors $\boldsymbol{s}_i^o$, $\boldsymbol{s}_j^o$.

Corollary: Consider the optimization problem defined in the Theorem and let us also include a set of linear equality constraints of the form $B^T\boldsymbol{t} = \boldsymbol{0}$, for some orthogonal matrix B. Then the new optimization problem can be reduced to the one defined in the Theorem.

Proof: Let P denote a matrix whose columns form a base for the null space of B, then $B^TP = 0$. It is known that any $\boldsymbol{t}$ satisfying the new set of constraints can be put under the form $\boldsymbol{t} = P\boldsymbol{x}$. Making this substitution in the problem we like to solve, we obtain for the variance $V(\boldsymbol{x}) = \boldsymbol{x}^TP^TQP\boldsymbol{x}$ and for the constraints $t_{0\cdots0}+\max_k\{\boldsymbol{x}^TP^T\boldsymbol{s}_k\} = 1, t_{0\cdots0}+\min_k\{\boldsymbol{x}^TP^T\boldsymbol{s}_k\} = -1$. Thus the optimum $\boldsymbol{x}$ can be obtained from the Theorem if we replace Q by P^TQP and $\boldsymbol{s}_k$ by $P^T\boldsymbol{s}_k$.

Comments: Using the Theorem or the Corollary we obtain the optimum transform. In order for our design to be complete we also need to specify the 1-D passband and stopband cutoff frequencies ω_p, ω_s. Following [11], we define

$$\cos(\omega_p) = \bar{F}_{C_p}, \cos(\omega_s) = \bar{F}_{C_s} \tag{24}$$

the two mean values of the transform on the two cutoff manifolds.

Using the definitions in (24) it is possible to force the actual cutoff manifolds (the points from the N-D frequency space that satisfy (4)) to pass through predefined points. This can be achieved by requiring the value of the transform at the prescribed points to be equal to the corresponding $\bar{F}_C$. It is easy to see that such a requirement generates equality constraints of the same form as the ones introduced in the Corollary. This will be used in the next section.

III. Application

We will apply our method for the design of a conic filter of angle $\theta = 20^\circ$ (passband cutoff manifold). The stopband cutoff manifold will be parallel to the passband manifold at a horizontal distance of 0.2π. We use $\alpha_p = 1$, $\alpha_s = 0$ meaning that we mainly concentrate in approximating the passband. Fig. 1 depicts the result of our method for the passband manifold. We can see that part of the stopband region enters the passband. This happens because with the proposed design method we are concerned only in approximating the manifolds in the best possible way without paying any attention in the correspondence of the regions. Fortunately for the class of filters under consideration, whenever this problem occurs, it can be easily corrected. Notice that the problem comes either from the fact that some passband corner has transform value smaller than $\cos(\omega_s)$ (thus entering the stopband) or some corner in the stopband region has transform value larger than $\cos(\omega_p)$ (thus entering the passband). In the first case we must require the transform value at the problematic corner to be equal to $\cos(\omega_p)$, while in the second case to be equal to $\cos(\omega_s)$. Both cases lead to equality constraints that are similar to the ones introduced in the Corollary. We use this idea for our example. We also require the origin to lie on the actual passband manifold. We thus introduce two new equality constraints. Solving the corresponding optimization problem yields the passband cutoff manifold of Fig. 2 with the problem corrected.

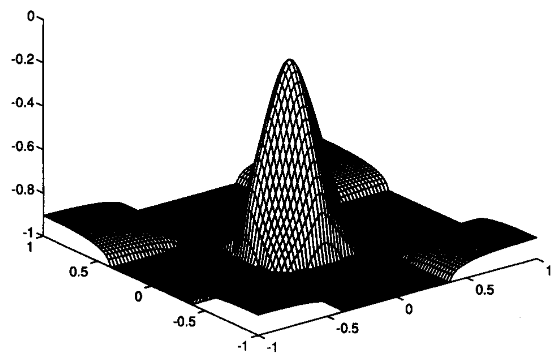

Fig. 1. Passband cutoff manifold of a cone filter with $\theta = 20^\circ$ and no additional constraints.

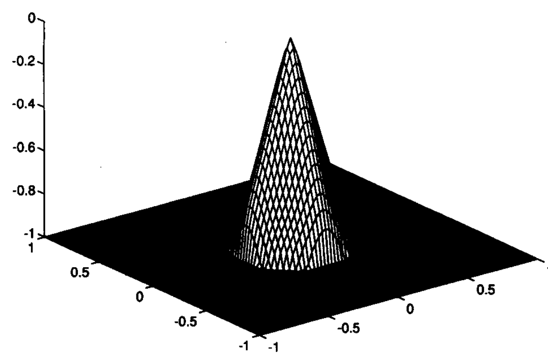

Fig. 2. Passband cutoff manifold of a cone filter with $\theta = 20^\circ$ and equality constraints on the origin and the corners.

IV. Conclusion

In this paper we extend an existing 2-D filter design method to the N-D case. With the proposed method we can design N-D hyperquadrantally symmetric FIR filters using the N-D first order symmetric McClellan transform. The optimum transform coefficients are obtained by solving a well defined constraint optimization problem. A closed form solution to the optimization problem is given (not having a counterpart in the 2-D case) and it consists in searching among a finite number of possibilities so as to obtain the optimum transform.

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Numerical Computation of the Cross-Covariance Sequences of Two-Dimensional Filters and Systems

Tong-Yi Guo and Chyi Hwang

***Abstract*—An effective numerical approach is presented for computing the general two-dimensional (2-D) complex integrals arising in the evaluation of cross-covariance sequences of 2-D digital systems. It converts the problem into the solution of a first-order differential equation with its function evaluations being computations of 1-D complex integrals, which can be efficiently accomplished by applying the complex version of the Euclid algorithm. A accurate solution can be obtained if an numerical integration scheme capable of accuracy control is used. To demonstrate the effectiveness of the presented approach, three numerical examples are worked out.**

I. Introduction

In the analysis and synthesis of causal 2-D linear discrete systems, it is often required to compute the complex integrals of the form [1]–[4]

$$J_{mn} = \frac{1}{(2\pi i)^2} \oint_{|z_1|=1} \oint_{|z_2|=1} G(z_1, z_2) H(z_1^{-1}, z_2^{-1}) \times z_1^{m-1} z_2^{n-1} \frac{dz_1 dz_2}{z_1 z_2}, \quad i = \sqrt{-1} \tag{1}$$

$$= \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} g_{j,k} h_{j-m,k-n} \tag{2}$$

where $h_{j,k}$ and $g_{j,k}, j, k = 0, 1, \cdots$, are the impulse response sequences of the causal stable transfer functions $G(z_1, z_2)$ and $H(z_1, z_2)$, respectively. For general 2-D discrete systems, denominator polynomials of $G(z_1, z_2)$ and $H(z_1, z_2)$ are not separable. The lack of a factorization theorem for 2-D polynomials makes the analytical computation of the complex integral J_{mn} a very difficult task. The closed-form solution for the integral J_{mn} is only available [5] for simple and low-order systems. The method presented in [1] and [6] for parametrically evaluating the integral J_{mn} is based on regarding $G(z_1, z_2)$ and $H(z_1, z_2)$ as 1-D transfer functions in z_2 (z_1, respectively) with coefficients being polynomials in z_1 (z_2, respectively). This method inherently involves the solution of a set of linear equations whose coefficients are polynomials in z_1 (z_2, respectively), which is tedious if the system orders are high. Recently, Lu *et al.* [7] have proposed a different approach to reduce the problem of computing 2-D complex integral J_{mn} into that of computing 1-D complex integral with respect to z_1 (z_2, respectively) by regarding z_2 (z_1, respectively) as a parameter and then using the recursive Å-J-A algorithm [8] to compute the resultant 1-D complex integrals. Although this approach avoids the solution of a parametric linear system, the iterative operation of polynomials in the first stage of performing Å-J-A algorithm will result in high-order 1-D rational transfer functions which may further cause numerical instability in the last stage of parametrically evaluating the resultant 1-D complex integrals.

Besides the above-mentioned parametric methods, numerical approaches are often used to compute the 2-D complex integral J_{mn}. The most commonly used numerical approach is the direct evaluation method [2], [3] which compute double sums in (2) with a finite number of terms. Although this method works well for many cases, the solution accuracy and the computational burden depends heavily on the number of terms used. Recently, a numerical approach [4] using 1–D rectangle-rule approximation and an algorithm [11] for 1–D complex integral has been presented to evaluate the 2–D complex integral J_{mn}. Like the direct evaluation method [2], [3] it cannot efficiently assure the solution accuracy.

The purpose of this brief is to present an effective numerical approach to the evaluation of the 2-D complex integral J_{mn}. With the same idea of Premaratne *et al.* [4], the problem of computing a 2-D complex integral is converted into that of evaluating a 1-D definite integral with its integrand being a 1–D parametric complex integral. However, the evaluation of a definite integral in this paper is accomplished through solving a first-order differential equation with a numerical integration scheme capable of automatic step-size adjustment to meet the specified accuracy requirement. Moreover, the function evaluations, which are computations of 1-D complex integrals, for the solution of differential equation are performed with the complex version of the Euclid algorithm. It is noted that the approach to accurate computation of a definite integral by solving a differential equation was applied successfully to compute 1-D complex integrals whose integrand contains irrational transfer functions [9].

II. Numerical Computation of General 2-D Complex Integrals

Let $z_2 = e^{i\theta_2}$, the 2-D complex integral J_{mn} in (1) can be written as

$$J_{mn} = \frac{1}{\pi} \int_0^{\pi} \left[\frac{1}{2\pi i} \oint_{|z_1|=1} G(z_1, e^{i\theta_2}) H(z_1^{-1}, e^{-i\theta_2}) z_1^m \frac{dz_1}{z_1} \right] \times e^{in\theta_2}\, d\theta_2 \tag{3}$$

$$= \frac{1}{\pi} \int_0^{\pi} J_m(\theta_2) e^{in\theta_2}\, d\theta_2. \tag{4}$$

Hence, the 2-D complex integral J_{mn} can be regarded as a 1-D definite integral whose integrand is the 1-D parametric complex

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