

## OPTIMAL STOPPING TIMES FOR DETECTING CHANGES IN DISTRIBUTIONS

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It is shown that Page's stopping time is optimal for the detection of changes in distributions, in a well defined sense. This work is a generalization of an existing result where it was shown that Page's stopping time is optimal asymptotically.

**1. Introduction.** Let us assume that  $X_1, X_2, \dots$  are independent and identically distributed random variables that are observed sequentially. Let also  $X_1, \dots, X_{m-1}$  have distribution function  $F_0$  and  $X_m, X_{m+1}, \dots$  distribution function  $F_1 \neq F_0$ . The two distributions are known, but the time of change  $m$  is assumed unknown. We are interested in finding stopping times that will detect the change with a delay as small as possible. Let  $P_m$  denote the true distribution of  $X_1, X_2, \dots$  when the change occurs at  $m$  and  $E_m$  the expectation under this distribution. We allow  $m$  to take the value infinity, denoting by this the case where no change occurs. Let  $\mathcal{F}_n$ ,  $n \geq 1$ , be the  $\sigma$ -algebra generated by  $\{X_1, X_2, \dots, X_n\}$ . We also consider an auxiliary  $\sigma$ -algebra  $\mathcal{Y}$  and we will assume that we can extend the measures  $P_m$  on  $\mathcal{Y}$  in such a way that for every real  $p \in [0, 1]$  we can generate an event in  $\mathcal{Y}$  that has probability  $p$  for every  $P_m$ . Define  $\mathcal{Y}_0 = \mathcal{Y}$  and  $\mathcal{Y}_n = \mathcal{F}_n \cup \mathcal{Y}$ . Allowable stopping times (s.t.) will be all those s.t. that have the form  $N = 0$  with probability  $(1 - p)$  and  $N = N'$  with probability  $p$  where  $N'$  is any s.t. measurable with respect to the filtration  $\{\mathcal{F}_n\}_{n \geq 1}$  and with the randomization  $p$  being done before any observation is taken.

We define optimality of a s.t. in the sense of Lorden [3]. That is, if  $N$  is a s.t. define

$$(1) \quad D_m(N) = \text{ess sup } E_m\{[N - m + 1]^+ / \mathcal{Y}_{m-1}\}, \quad m \geq 1,$$

$$(2) \quad D(N) = \sup_{m \geq 1} D_m(N).$$

Thus we consider the conditional expectation of the delay over those events before the change occurs that least favor the detection of the change. We would like to minimize  $D(N)$  over those s.t. from the allowable class that satisfy the following constraint on the rate of false detections:

$$(3) \quad E_\infty\{N\} \geq \gamma > 0.$$

Our goal in the next section will be to prove that Page's s.t. is optimal in the above sense. Let us first define this s.t. For simplicity we will assume that  $F_0$  and  $F_1$  are mutually absolutely continuous. Let  $l(x)$  denote the Radon–Nikodym

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derivative of  $F_1$  with respect to  $F_0$ . We will assume that  $l(X_1)$  has no atoms with respect to  $P_\infty$ . Let us define the following sequence of random variables:

$$(4) \quad \begin{aligned} S_0 &= 0, \\ S_n &= \max\{S_{n-1}, 1\}l(X_n), \quad n \geq 1. \end{aligned}$$

We define Page's s.t.  $N_p$  as follows:

$$(5) \quad N_p = \inf\{n \geq 1: S_n \geq \mu\},$$

where  $\mu$  is a nonnegative constant and the infimum of the empty set is infinity. Page's s.t. is defined a little differently here than it is usually defined in the literature.  $N_p$  is usually defined as the first  $n$  for which  $T_n = \max\{S_n, 1\}$  exceeds  $\mu$ . The two definitions are equivalent when  $\mu > 1$ , but there is a difference when  $\mu < 1$ . When  $\mu < 1$ , with the definition that uses  $T_n$  we stop at  $n = 1$ , but with the definition in (5) this is not the case. As we will see in the next section there exists a nontrivial range of values of  $\gamma$  for which  $\mu \in (0, 1]$ . With the following lemma we give some properties of the sequences  $S_n$  and  $T_n$  that will be used later.

**LEMMA 1.** *Let  $T_n = \max\{S_n, 1\}$ . For any  $n > m \geq 1$  and for fixed  $\{X_{m+1}, \dots, X_n\}$ , the quantity  $S_n$  is a nondecreasing function of  $T_m$ . Also  $T_n$  can be written as*

$$(6) \quad T_n = \sum_{j=1}^{n+1} [1 - S_{j-1}]^+ \prod_{k=j}^n l(X_k),$$

where we define  $\prod_{k+1}^k = 1$ .

**PROOF.** The property that  $S_n$  is a nondecreasing function of  $T_m$  (for fixed  $\{X_{m+1}, \dots, X_n\}$ ) can be proved by induction and using the definition in (4). To prove (6), we can see from (4) that

$$(7) \quad T_n = \max\{S_n, 1\} = S_n + [1 - S_n]^+.$$

If we use (4), (7) and induction we can easily show (6).  $\square$

A very important consequence of the monotonicity of  $S_n$  with respect to  $T_m$  is that on the event  $\{N_p \geq m\}$  the s.t.  $N_p$  is nonincreasing with  $T_{m-1}$ ; thus, the essential supremum in (1) is achieved for  $T_{m-1} = 1$  (or  $S_{m-1} \leq 1$ ). This means that restarting Page's s.t. at  $m$  gives the worst conditional average delay and thus from stationarity all  $D_m(N_p)$  are equal. The s.t.  $N_p$  is thus an equalizer rule, a very important property for proving its optimality.

**2. Optimal stopping time.** Notice first that for  $\gamma > 0$  we have  $D(N) \geq 1$ . This is true because with  $E_\infty\{N\} \geq \gamma > 0$  it is not possible to stop a.s. at  $n = 0$  and thus we will take at least one sample. With this remark we have that for  $1 \geq \gamma > 0$ , the optimal s.t. (say  $N_0$ ) is  $\{\text{stop at } n = 0 \text{ with probability } 1 - \gamma \text{ otherwise stop at } n = 1\}$ . This yields  $D(N_0) = 1$  and  $E_\infty\{N_0\} = \gamma$ . We now consider the case  $1 < \gamma < \infty$ . With the next lemma we will show that in order to find the optimal s.t. it is enough to limit ourselves to a smaller class of s.t.

LEMMA 2. *In order to minimize  $D(N)$  over the s.t. that satisfy (3) it is enough to consider those s.t. that satisfy (3) with equality.*

PROOF. From the way we defined our class of s.t., we have that  $N = 0$  with probability  $(1 - p)$  and  $N = N'$  with probability  $p$ , where  $N'$  is measurable with respect to  $\{\mathcal{F}_n\}$ . If  $p > 0$  then we have that  $D_m(N) = D_m(N')$  for every  $m > 1$  and thus  $D(N) = D(N')$ . Notice also that  $E_\infty\{N\} = pE_\infty\{N'\}$ , where the product  $pE_\infty\{N'\}$  is defined as being zero when  $p = 0$ . If  $E_\infty\{N\} = \infty$  then  $p > 0$  and  $E_\infty\{N'\} = \infty$ . We can always find a large enough integer  $K$  such that if we define  $M' = \min\{N', K\}$  to have  $pE_\infty\{M'\} > \gamma$ . If now  $M$  is the randomization of  $M'$  with probability  $p$  then  $D(M) = D(M') \leq D(N') = D(N)$ . It is thus enough to consider s.t. that have finite  $E_\infty\{N\}$ . Let now  $\infty > E_\infty\{N\} = pE_\infty\{N'\} > \gamma$ ; this means  $p > 0$  and  $E_\infty\{N'\} < \infty$ . By defining a new s.t.  $M$  that is equal to  $N'$  with randomization probability  $p' = \gamma/E_\infty\{N'\} < p$  we have  $E_\infty\{M\} = \gamma$  and  $D(M) = D(N') = D(N)$ . This concludes the proof.  $\square$

In the following lemma we introduce a lower bound for  $D(N)$  that we will use as our performance measure instead of  $D(N)$ .

LEMMA 3. *For any s.t.  $N$  satisfying  $0 < E_\infty\{N\} < \infty$  we have that*

$$(8) \quad D(N) \geq \frac{E_\infty\left\{\sum_{k=0}^{N-1} \max\{S_k, 1\}\right\}}{E_\infty\left\{\sum_{k=0}^{N-1} [1 - S_k]^+\right\}} = \bar{D}(N),$$

where we define  $\sum_{k=j}^{j-1} = 0$ . We have equality in (8) when  $N = N_p$ .

PROOF. Let  $I(A)$  denote the indicator function of the event  $A$ . Define

$$(9) \quad B_m(N) = E_m\left\{[N - m + 1]^+ / \mathcal{Y}_{m-1}\right\}.$$

Because the event  $\{N \geq k\}$  is  $\mathcal{Y}_{k-1}$  measurable we have

$$(10) \quad \begin{aligned} B_m(N) &= \sum_{k=m}^{\infty} E_m\{I(N \geq k) / \mathcal{Y}_{m-1}\} \\ &= \sum_{k=m}^{\infty} E_\infty\left\{\left[\prod_{j=m}^{k-1} I(X_j)\right] I(N \geq k) / \mathcal{Y}_{m-1}\right\} \\ &= E_\infty\left\{\left[\sum_{k=m}^N \prod_{j=m}^{k-1} I(X_j)\right] / \mathcal{Y}_{m-1}\right\}. \end{aligned}$$

Notice that in (1)  $D_m(N)$  was defined as the essential supremum of  $B_m(N)$ . Since for every  $m > 1$  we have  $D(N) \geq D_m(N)$  using (10), this yields

$$(11) \quad \begin{aligned} &E_\infty\{[1 - S_{m-1}]^+ I(N \geq m)\} D(N) \\ &\geq E_\infty\{[1 - S_{m-1}]^+ I(N \geq m) B_m(N)\} \\ &= E_\infty\left\{I(N \geq m) \sum_{k=m}^N [1 - S_{m-1}]^+ \prod_{j=m}^{k-1} I(X_j)\right\}. \end{aligned}$$

When  $N = N_p$  we have equality in (11). This is true because  $D(N_p) = D_m(N_p)$  for every  $m$  and, because as we said in the introduction, the essential supremum of  $B_m(N_p)$  is achieved on the event  $\{N_p \geq m\} \cap \{S_{m-1} \leq 1\}$  and that  $[1 - S_{m-1}]^+ I(N_p \geq m)$  may be nonzero only on this event. Summing now (11) for all  $m \geq 1$ , interchanging summations and expectations and using (6), the last term of the inequality in (11) gives

$$\begin{aligned}
 & \sum_{m=1}^{\infty} E_{\infty} \left\{ I(N \geq m) \sum_{k=m}^N [1 - S_{m-1}]^+ \prod_{j=m}^{k-1} l(X_j) \right\} \\
 &= E_{\infty} \left\{ \sum_{m=1}^N \sum_{k=m}^N [1 - S_{m-1}]^+ \prod_{j=m}^{k-1} l(X_j) \right\} \\
 (12) \quad &= E_{\infty} \left\{ \sum_{k=1}^N \left[ \sum_{m=1}^k [1 - S_{m-1}]^+ \prod_{j=m}^{k-1} l(X_j) \right] \right\} \\
 &= E_{\infty} \left\{ \sum_{k=1}^N T_{k-1} \right\} = E_{\infty} \left\{ \sum_{k=0}^{N-1} T_k \right\} = E_{\infty} \left\{ \sum_{k=0}^{N-1} \max\{S_k, 1\} \right\}.
 \end{aligned}$$

For the first term in (11) we have that

$$(13) \quad \sum_{m=1}^{\infty} E_{\infty} \{ I(N \geq m) [1 - S_{m-1}]^+ \} = E_{\infty} \left\{ \sum_{m=0}^{N-1} [1 - S_m]^+ \right\}.$$

The quantity in (13) is less than  $E_{\infty}\{N\}$ , thus finite. For  $N \geq 1$  we also have that

$$(14) \quad \sum_{m=0}^{N-1} [1 - S_m]^+ \geq 1;$$

thus, the quantity in (14) is no less than the probability  $P_{\infty}\{N \geq 1\}$ , which is nonzero since by assumption we have  $E_{\infty}\{N\} > 0$ . Thus, we have shown (8).  $\square$

In order now to show that Page's s.t. is optimal it is enough to show that among all s.t. that satisfy  $E_{\infty}\{N\} = \gamma$ , Page's s.t. is the one that minimizes  $\bar{D}(N)$ . Specifically  $N_p$  minimizes  $\bar{D}(N)$  by simultaneously minimizing its numerator and maximizing its denominator. One can now see why it was necessary to limit ourselves to the class  $E_{\infty}\{N\} = \gamma$ . If we had condition (3) instead, this gives  $N = \infty$  as optimal for the denominator. Since from now on all events will be considered with respect to the measure  $P_{\infty}$ , for simplicity we drop the subscript  $\infty$ . With the following theorem we show that Page's s.t. is optimal for a whole class of optimization problems.

**THEOREM 1.** *Let  $\infty > \gamma > 1$ . If  $\varphi(z)$  is a continuous nonincreasing function, well defined for all  $z \geq 0$  with  $\varphi(0)$  bounded, then Page's s.t. satisfies*

$$(15) \quad \sup_N E \left\{ \sum_{k=0}^{N-1} \varphi(S_k) \right\} = E \left\{ \sum_{k=0}^{N_p-1} \varphi(S_k) \right\}$$

for all s.t.  $N$  that satisfy  $E\{N\} = \gamma$ .

Using Theorem 1 we can easily show that  $N_p$  minimizes  $\bar{D}(N)$ . With  $\varphi(z) = -\max\{z, 1\}$ , Theorem 1 shows that  $N_p$  minimizes the numerator of  $\bar{D}(N)$  and, with  $\varphi(z) = [1 - z]^+$ , that it maximizes the denominator. By assumption we have that  $\varphi(z)$  is bounded from above by  $\varphi(0) < \infty$ . Without loss of generality we can assume that  $\varphi(z)$  is also bounded from below. This is true because if  $\varphi(z)$  is not bounded from below we can always define a new function  $\varphi'(z) = \max\{\varphi(z), A\}$ , where  $A \leq \varphi(\mu)$  ( $\mu$  is the threshold for  $N_p$ ). We will thus have

$$(16) \quad E\left\{\sum_{k=0}^{N-1} \varphi(S_k)\right\} \leq E\left\{\sum_{k=0}^{N-1} \varphi'(S_k)\right\}.$$

Notice that we have equality in (16) when  $N = N_p$ . This is true because the two sums in (16) are taken up to  $N - 1$ ; thus, for  $N = N_p$ ,  $S_k$  is always in the region where  $\varphi(S_k) = \varphi'(S_k)$ . Clearly if  $N_p$  maximizes the right-hand side of (16), it also maximizes the left-hand side. We will thus assume that  $\varphi(z)$  is bounded and let  $D < \infty$  be a bound.

Before going to the proof of Theorem 1 we give some definitions and present certain results that will be useful for this proof. Let us denote by  $\nu_r$  the time of the  $r$ th entry of  $S_n$  in the set  $[0, 1]$ , i.e.,  $\nu_0 = 0$  and, for  $r \geq 1$ ,

$$(17) \quad \nu_r = \inf\{n > \nu_{r-1} : S_n \leq 1\}.$$

In the next lemma we present a property of the time  $\nu_1$ .

LEMMA 4. *All finite moments of the time  $\nu_1$  exist.*

PROOF. Since for  $a \geq 0$  the function  $a^s$  is a convex function of  $s$  we have for  $0 \leq s \leq 1$  that  $a^s \leq (1 - s) + sa$ , with equality if and only if  $a = 1$ . If now  $a = l(X_1)$  and we define  $\alpha(s) = E\{[l(X_1)]^s\}$ , we conclude that  $\alpha(s) \leq 1$ . Since  $l(X_1)$  is not a constant equal to unity, there exists  $s_0$  that satisfies  $\alpha(s_0) < 1$ . Notice now that

$$\begin{aligned} P\{\nu_1 = k\} &\leq P\{S_{k-1} > 1\} = P\left\{\prod_{j=1}^{k-1} l(X_j) > 1\right\} \\ &= P\left\{\prod_{j=1}^{k-1} [l(X_j)]^{s_0} > 1\right\} \leq [\alpha(s_0)]^{k-1}. \end{aligned}$$

To show that all the moments exist, we have

$$E\{\nu_1^j\} = \sum_{k=1}^{\infty} k^j P\{\nu_1 = k\} \leq \sum_{k=1}^{\infty} k^j [\alpha(s_0)]^{k-1} < \infty,$$

and this concludes the proof.  $\square$

In order to solve the constrained optimization problem defined in (15) using the Lagrange multiplier technique, we will reduce it to an unconstrained optimization problem. Let  $S_0 = x \geq 0$  and, for any real  $\lambda$  define

$$(18) \quad V(x, \lambda) = \sup_N E\left\{\sum_{k=0}^{N-1} [\varphi(S_k) - \lambda]\right\},$$

where now the supremum is taken over all s.t.  $N$ . The function  $V(x, \lambda)$  is nonnegative and can take the value infinity. We are interested in finding for which values of  $x$  and  $\lambda$   $V(x, \lambda)$  is finite. With the next lemma we can see the behavior of  $V(x, \lambda)$  with respect to  $x$  when  $\lambda$  is fixed.

**LEMMA 5.** *Let  $V(x, \lambda)$  be defined by (18); then  $V(x, \lambda)$  is finite if and only if  $V(0, \lambda)$  is finite.*

**PROOF.** Notice first that using Lemma 1 and the monotonicity of  $\varphi(z)$  we conclude that  $V(x, \lambda)$  is nonincreasing in  $x$  for fixed  $\lambda$ . With this property the if part is easy to show since  $V(x, \lambda) \leq V(0, \lambda) < \infty$ . For the only if part now, assume that  $V(x, \lambda) < \infty$  for some  $x = x_0$ . For  $x \geq x_0$  we then have  $V(x, \lambda) \leq V(x_0, \lambda) < \infty$ . We will now show that we also have  $V(0, \lambda) < \infty$ . Let  $N_0$  denote Page's s.t. with threshold  $x_0$ . For any s.t.  $N$  define  $N' = \min\{N, N_0\}$ ; then

$$\begin{aligned} & E \left\{ \sum_{k=0}^{N-1} [\varphi(S_k) - \lambda] \right\} \\ &= E \left\{ \sum_{k=0}^{N'-1} [\varphi(S_k) - \lambda] \right\} + E \left\{ \sum_{k=N'}^{N-1} [\varphi(S_k) - \lambda] \right\} \\ &\leq (D + |\lambda|)E\{N_0\} + \sum_{j=1}^{\infty} \sup_{N \geq j} E \left\{ \sum_{k=j}^{N-1} [\varphi(S_k) - \lambda] I(N_0 = j) \right\} \\ &\leq (D + |\lambda|)E\{N_0\} + V(x_0, \lambda) < \infty \end{aligned}$$

and this concludes the proof.  $\square$

From this lemma we have that for fixed  $\lambda$ ,  $V(x, \lambda)$  is either finite for all values of  $x$  or it is infinite. With the following theorem we identify the range of values of  $\lambda$  for which  $V(0, \lambda)$  (and thus  $V(x, \lambda)$ ) is finite.

**THEOREM 2.** *Let  $\xi_1 = \sum_{k=1}^{\nu_1} \varphi(S_k)$  and  $\lambda_0 = E\{\xi_1\}/E\{\nu_1\}$ .*

(a) *If  $\infty > \lambda > \lambda_0$  and  $S_0 = x \geq 0$ , then we have*

$$E \left\{ \sup_n \left[ \sum_{k=0}^n [\varphi(S_k) - \lambda] \right]^+ \right\} < \infty.$$

(b) *If  $\xi_1 - \lambda_0 \nu_1$  is not a constant, then  $V(0, \lambda) \rightarrow \infty$  as  $\lambda \rightarrow \lambda_0 +$ .*

**PROOF.** Condition (a) is sufficient to ensure existence of  $V(x, \lambda)$  ([6], page 69). First notice that  $|E\{\xi_1\}| \geq DE\{\nu_1\}$  (where  $D$  is a bound for  $\varphi(z)$ ), thus  $\lambda_0$  is finite. To show (a) for every  $x \geq 0$ , it is enough to show it (using Lemma 1) for  $x = 0$ . Consider the sequence  $\nu_1, \nu_2, \dots$  defined in (17); it goes to infinity a.s. Define  $\xi_r = \sum_{\nu_{r-1}+1}^{\nu_r} \varphi(S_k)$  and  $\eta_r = \nu_r - \nu_{r-1}$ . Using the strong Markov property of the sequence  $\{S_n\}$ , we have that the two sequences  $\{\xi_r\}$  and  $\{\eta_r\}$  are i.i.d. sequences of random variables. Let now  $\nu_{r-1} < n \leq \nu_r$ ; then,

$$(19) \quad \sum_{k=0}^n [\varphi(S_k) - \lambda] \leq \sum_{k=1}^r [\xi_k - \lambda \eta_k] + \eta_r D'$$

where  $D' = D + |\lambda|$ . The sequence  $\{\omega_k\}$  with  $\omega_k = \xi_k - \lambda\eta_k$  is also an i.i.d. sequence. For  $\lambda > \lambda_0$  we have that  $E\{\omega_1\} < 0$ . Let  $\delta > 0$  be such that  $E\{\omega_1\} + \delta < 0$ ; then, from (19) we have

$$(20) \quad \begin{aligned} \left[ \sum_{k=0}^n [\varphi(S_k) - \lambda] \right]^+ &\leq \left[ \sum_{k=1}^r [\omega_k + \delta] \right]^+ + [\eta_r D' - \delta r]^+ \\ &\leq \left[ \sum_{k=1}^r [\omega_k + \delta] \right]^+ + D' \eta_r I(\eta_n \geq \delta' r), \end{aligned}$$

where  $\delta' = \delta/D'$ . From (20) we have

$$(21) \quad \begin{aligned} E \left\{ \sup_n \left[ \sum_{k=0}^n [\varphi(S_k) - \lambda] \right]^+ \right\} \\ \leq E \left\{ \sup_r \left[ \sum_{k=1}^r [\omega_k + \delta] \right]^+ \right\} + D' \sum_{r=1}^{\infty} E \{ \eta_r I(\eta_r \geq \delta' r) \}. \end{aligned}$$

For the first term in (21) to be finite, sufficient conditions are  $E\{\omega_1\} + \delta < 0$  and  $E\{([\omega_1 - E\{\omega_1\}]^+)^2\} < \infty$  ([2], page 92). The first condition is satisfied; we can easily show that the second is also satisfied using the properties that  $\varphi(z)$  is bounded, that the second moment of  $\nu_1$  is finite (Lemma 4) and the fact that for any real  $a$  we have  $(a^+)^2 \leq a^2$ . To show that the second term in (21) is finite notice that since  $\eta_r$  is distributed as  $\nu_1$  we have

$$\sum_{r=1}^{\infty} E \{ \eta_r I(\eta_r \geq \delta' r) \} \leq \frac{E\{\nu_1^3\}}{(\delta')^2} \sum_{r=1}^{\infty} \frac{1}{r^2} < \infty.$$

To prove (b), notice first that  $V(0, \lambda)$  is nonincreasing in  $\lambda$ ; thus, the limit of  $V(0, \lambda)$  exists (being possibly infinity). Consider now s.t. that can stop only at the instances  $\nu_r$  and the decision whether to stop or continue at  $\nu_r$  is made by using the random variables  $\xi_1, \dots, \xi_r$  and  $\eta_1, \dots, \eta_r$ . For this case we have

$$\sum_{k=0}^{\nu_R-1} [\varphi(S_k) - \lambda] \geq \sum_{k=1}^R [\xi_k - \lambda\eta_k] - 2D'.$$

This yields

$$(22) \quad V(0, \lambda) \geq E \left\{ \sum_{k=1}^R [\xi_k - \lambda\eta_k] \right\} - 2D'.$$

Consider now a s.t.  $R$  with  $E\{R\} < \infty$ , from (22) taking limits with respect to  $\lambda$  gives

$$(23) \quad \lim_{\lambda \rightarrow \lambda_0^+} V(0, \lambda) \geq E \left\{ \sum_{k=1}^R [\xi_k - \lambda_0 \eta_k] \right\} - 2D'.$$

The random variable  $\xi_1 - \lambda_0 \eta_1$  has zero mean; thus, from [2, page 27], the right-hand side in (23) can be made arbitrarily large. This concludes the proof of Theorem 2.  $\square$

We are now ready to prove Theorem 1.

**PROOF OF THEOREM 1.** Notice that the first term in the sum in (15) is  $\varphi(S_0) = \varphi(0)$ . In general, the value  $\varphi(0)$  is not attainable by  $\varphi(S_k)$  for  $k > 0$  because  $\varphi(S_k) \leq \varphi(l(X_k)) \leq \varphi(z_0)$ , where  $z_0 = \text{ess inf}\{l(X_1)\}$ . This will be, for example, the case when  $\varphi(z) = [1 - z]^+$  and  $z_0 > 0$ . The quantity  $\varphi(z_0)$  will play a role in our proof. We distinguish two cases

(1)  $\varphi(z_0) = \varphi(\mu)$ . For this case we have

$$E\left\{\sum_{k=0}^{N-1} \varphi(S_k)\right\} \leq [\varphi(0) - \varphi(z_0)]P\{N > 0\} + \varphi(z_0)E\{N\} \\ \leq [\varphi(0) - \varphi(z_0)] + \varphi(z_0)\gamma,$$

with equality when  $N = N_p$  because for  $k > 0$  and  $z_0 \leq S_k \leq \mu$  we have  $\varphi(S_k) = \varphi(z_0)$ .

(2)  $\varphi(z_0) > \varphi(\mu)$ . Let  $\lambda > \lambda_0$ . For every s.t.  $N$ ,  $E\{\sum_{k=0}^{N-1} [\varphi(S_k) - \lambda]\}$  is linear in  $\lambda$ ; thus,  $V(x, \lambda)$  being the supremum over  $N$  of this expression, is a convex function of  $\lambda$ . If we denote  $[x] = \max\{x, 1\}$ , the function  $E\{V([x]l(X_1), \lambda)\}$  will also be convex and thus continuous in  $\lambda$ .

Consider now the equation

$$\beta(\lambda) = \varphi(\mu) - \lambda + E\{V([\mu]l(X_1), \lambda)\} = 0.$$

The function  $\beta(\lambda)$  is continuous in  $\lambda$ . Since  $V(x, \varphi(0)) = 0$ , we have  $\beta(\varphi(0)) = \varphi(\mu) - \varphi(0) < 0$  and  $\beta(\lambda) \rightarrow \infty$  as  $\lambda \rightarrow \lambda_0 +$  (Theorem 2(b)). Thus, there exists a  $\lambda^*$  with  $\varphi(0) > \lambda^* > \lambda_0$  that satisfies  $\beta(\lambda^*) = 0$ . Let now  $\lambda = \lambda^*$ . From Theorem 2(a) and [6, page 69] we have that  $V(x, \lambda^*)$  exists and satisfies

$$V(x, \lambda^*) = [\varphi(x) - \lambda^* + E\{V([x]l(X_1), \lambda^*)\}]^+.$$

From [6, page 74] we have that the optimum is to stop when  $V(S_n, \lambda^*) = 0$ . Since  $\varphi(x)$  and  $E\{V([x]l(X_1), \lambda^*)\}$  are nonincreasing functions of  $x$ , this will be the case when  $S_n \geq \mu$ , i.e., Page's s.t.  $N_p$ .  $\square$

**COMMENTS.** It is very difficult in general to relate explicitly  $\gamma$  to  $\mu$ , though there is a range of values of  $\gamma$  where this is possible. Let us consider the case  $\mu \leq 1$ . For this case  $N_p$  is equivalent to  $N_p = \inf\{n: l(X_n) \geq \mu\}$ . In other words, given that there is no stop before  $n$ , we have that  $S_k \leq 1$  for  $k < n$ . Indeed if for some  $k$  we had  $S_k > 1$ , then we would also have  $S_k > \mu$ , thus having a stop at  $k$ , a contradiction. For this case the expectation of  $N_p$  under  $P_1$  and  $P_\infty$  is  $E_i\{N_p\} = [P_i\{l(X_1) \geq \mu\}]^{-1}$ ,  $i = 0, \infty$ . Thus, for

$$1 < \gamma \leq [P_\infty\{l(X_1) \geq 1\}]^{-1},$$

the relation between  $\gamma$  and  $\mu$  is given by  $P_\infty\{l(X_1) \geq \mu\} = \gamma^{-1}$ . For other values of  $\gamma$  the integral equation defined in Page's paper [4] can be used, but clearly this is a more complicated situation. For approximations see [7]. In the introduction we assumed that  $l(X_1)$  has no atoms. In the general case we have to modify



the s.t.  $N_p$  by including a randomization whether to continue or stop every time we have  $S_n = \mu$ .

The approach we have followed here is non-Bayesian. The criterion  $D(N)$  that we used takes into account only the worst possible situation before a change occurs; thus, it may be considered as conservative. In [5], following again a non-Bayesian approach, the criterion  $D(N) = \sup_m E_m\{(N - m)/N \geq m\}$  is used. The optimal s.t. obtained for this criterion is again of the form "stop when a statistic  $R_n$  exceeds a threshold." The statistic  $R_n$  satisfies the recursion  $R_n = (1 + R_{n-1})l(X_n)$ . Notice the similarity with the statistic  $S_n$  used for  $N_p$ ; both can be written in the form  $Z_n = \omega(Z_{n-1})l(X_n)$ , where  $\omega(z)$  is a univariate function satisfying  $\omega(z) \geq 1$ . Clearly  $S_n$  is more conservative with respect to past information (included in  $S_{n-1}$ ) because it does not take it into account if it is not important enough. This does not happen with  $R_n$ . In any case, Page's s.t. is very popular and widely used in practice because it has the important property of combining detection and estimation. Specifically, the largest  $n \leq N_p$  for which  $S_n \leq 1$  is the maximum likelihood estimate of the change time  $m$  using all observations up to time  $N_p$ . Finally for Bayesian approaches see [1] and [6, page 193].

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