Robust Wiener Filters for Imprecise

Second Order Statistics

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ABSTRACT

The problem of Wiener filtering is considered for the case where the cross spectral density matrix of the signal and noise is not precisely known. Filters are obtained which are saddle point solutions for the criterion of performance (mean square error) over the classes of allowable density matrices. Solutions for various classes are given.

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CHAPTER 1

Introduction.

The problem of linear estimation of signal in additive noise using as a criterion the mean square error (MSE) was solved by the classical works of Wiener and Kolmogorov. The optimum linear filter is completely specified if we know exactly the second order statistics of the signal and the noise. In most applications this assumption of precise knowledge is unrealistic. It is useful to have processes for design of filters that perform in a satisfactory way, when the second order statistics cannot be precisely specified. Several works have considered this problem. In all cases there exist a known class of second order statistics which contain the actual second order statistics of the signal and the noise. The broader this class is the more vague our knowledge is about the actual second order statistics .

Nahi and Weiss [1,2] derived the bounding filter H_b. This filter is a Wiener fielter (min MSE) for some bounding second order statistics. Its min MSE behaves as an upper bound to any other MSE resulting by applying H_b to any characteristic from the class of the allowable second order statistics. Usually the second order statistics that is used for the design of H_b does not belong to the class and the upper bound cannot be reached by any second order statistics from the class.

Kassam and Lim [3] inspired by Kuznetsov [4] and Huber [5] defined the robust filter H_r. This filter is again a Wiener filter for some second order statistics but now this second order statistics belongs to the given class. Its min MSE error behaves as an upper bound to any other MSE using H_r and this time the bound is a maximum since it can be reached by some second order statistics from the class. That is, the solution is a saddle point for the class. Poor [6,7,8] generalized these ideas and showed the similarity between the robust filtering and robust hypothesis testing. It is important to notice that the bounding filtering and the robust filtering approaches give the same results whenever in the bounding filtering case the nominal second order statistics that is used for the design of H_b belongs to the class. In the cases where a robust filter exists it has always equal or superior performance in terms of upper bounding the MSE.

In all the previous approaches [3,6,7,8] signal and noise were considered independent. In this work we apply the robust filtering idea to the correlated signal and noise case. We design filters for certain classes of second order statistics. The models for the classes that we will be considering are the band models (upper and/or lower bounds on the spectral densities). The whole approach is based on a theorem whose validity does not depend on the assumed models.

The presence of correlation is possible in many applications. An example is a multipath channel with a strong signal component , weak unwanted multipath signal and regular noise. The total "noise" , the unwanted part, is obviously correlated with the signal. We will give numerical examples to show how bad the performance can be if we design the filter assuming signal and noise uncorrelated,when they really are correlated.

In chapter 2 we set up the problem explicitly and we prove a general result. This is applied to specific cases in chapter 3. In chapter 4 we give some numerical examples and make some comparisons and in chapter 5 there is the conclusion and topics for further investigation.

CHAPTER 2

General Theory of Robust Filtering.

Before defining the robust filter we are going to summarize the Wiener filtering theory since we will be using all its results.

2.1 Wiener Filters.

Let us assume that our processes are real, jointly wide sense stationary and zero mean. If s(t) and n(t) are the signal and the noise processes then we can define the covariance matrix as :

$$R = \begin{bmatrix} R_{ss}(\tau) & R_{sn}(\tau) \\ R_{ns}(\tau) & R_{nn}(\tau) \end{bmatrix}$$

where

$$R_{ss}(\tau) = E[s(t).s(t-\tau)]$$

$$R_{sn}(\tau) = E[s(t).n(t-\tau)]$$

$$R_{nn}(\tau) = E[n(t).n(t-\tau)]$$

Fourier transforming we get the spectral density matrix

$$D = \begin{bmatrix} D_s(w) & D_{sn}(w) \\ D^*_{sn}(w) & D_n(w) \end{bmatrix}$$

Since we are going to use mostly the D matrix we now state the properties that characterize a matrix as a density matrix. These are :

i. $D_s(w), D_n(w)$ are even and nonnegative functions

ii. $|D_{sn}(w)| \leq \sqrt{D_s(w)\ D_n(w)}$ (I)

Thus D is a nonnegative definite matrix with diagonal elements even functions.

Given random processes s(t) and n(t) with correlation matrix R or density matrix D and a linear filter h(t) with Fourier transform H(w) , the MSE for signal estimation using this filter is :

$$e(D,H) = E[\ s(t) - \int_{\infty}^{\infty} h(v).x(t-v)dv\ ]^2$$

$$= R_{ss}(0) - 2\int_{\infty}^{\infty} h(v).R_{sx}(v)dv + \int\int_{\infty}^{\infty} h(v).h(u).R_{xx}(v-u)dv.du \qquad (2)$$

where x(t)=s(t)+n(t) is the received process. Using Fourier transforms and Parseval's theorem we can write (2) as follows :

$$e(D,H) = \frac{1}{2\pi}\int_{\infty}^{\infty}[\ D_s(w) - 2.H(w).D_{sx}^{*}(w) + |H(w)|^2.D_x(w)\ ]dw \qquad (3)$$

The optimum noncausal linear filter for the matrix D is the one that minimizes expression (3) and is given by

$$H_o(w) = \frac{D_{sx}(w)}{D_{xx}(w)} \qquad (4)$$

or in terms of signal and noise

$$H_o(w) = \frac{D_s(w) + D_{sn}(w)}{D_s(w)+D_n(w)+2R_e[D_{sn}(w)]} \tag{5}$$

The corresponding error is the optimum error for D and is given by :

$$e_{op}(D) = \frac{1}{2\pi}\int_{\infty}^{\infty} \frac{D_s(w).D_x(w) - |D_{sx}(w)|^2}{D_x(w)} \tag{6}$$

or in terms of signal and noise

$$e_{op}(D) = \frac{1}{2\pi}\int^{\infty} \frac{D_s(w).D_n(w) - |D_{sn}(w)|^2}{D_s(w)+D_n(w)+2.Re[D_{sn}(w)]} \tag{7}$$

2.2 Definition of Robust Filters.

Assume that a class Δ of density matrices is given then a robust filter H_r is defined by the following conditions

a. H_r is an optimum (Wiener) filter for some matrix $D^r \in \Delta$. This means :

$$e_{op}(D^r) = e(D^r,H_r) \leq e(D^r,H) \tag{8}$$

b. For any $D \in \Delta$ we have that

$$e(D,H_r) \leqslant e_{op}(D^r) \tag{9}$$

Combining a and b we have the saddle point relation

$$e(D,H_r) \leqslant e_{op}(D^r) = e(D^r,H_r) \leqslant e(D^r,H) \tag{I0}$$

for any $D \in \Delta$ and for any linear filter H. Any pair (D^r, H_r) that satisfies (10) is called a saddle point solution to our problem, for the given class Δ and the class of all linear filters. Obviously we need only D^r since from D^r by (4) we can find H_r. We will call D^r a <u>least favorable (l.f.) matrix</u> and with the following theorem we give a necessary and sufficient condition for a matrix to be l.f.

2.3 Theorem 0.

Let Δ be a convex class of density matrices. The pair (D^r, H_r) is a saddle point solution to our problem for the given class Δ and the class of all linear filters if and only if :

$$e_{op}(D^r) = \max_{D \in \Delta} e_{op}(D) \tag{II}$$

The proof is given in the appendix.

<u>Comments on Theorem 0.</u>

In theorem 0 is stated that the l.f. matrix maximizes the $e_{op}(D)$ over the class Δ. We can find conditions for the class Δ to assure the existence of the max $e_{op}(D)$. Such condition could be that Δ is closed under a suitable metric so that the image of Δ under the transformation $e_{op}(D)$ is also a closed subset of the real numbers. Also either of the powers of signal and noise has to be bounded, so that the image of Δ is also bounded.

As we can see from theorem 0 the existence of max $e_{op}(D)$ is a necessary condition for the existence of the saddle point solution. But a necessary condition for existence translates into a sufficient condition for non-existence. This means that if $\max e_{op} = \infty$ or $\sup e_{op}(D) < \infty$ but there exist no maximum, then there is no saddle point.

The convexity of Δ guarantees the sufficiency part of theorem 0. It is not a necessary condition for the existence of the saddle point solution. In figure 1 there is a simple example that illustrates this statement.

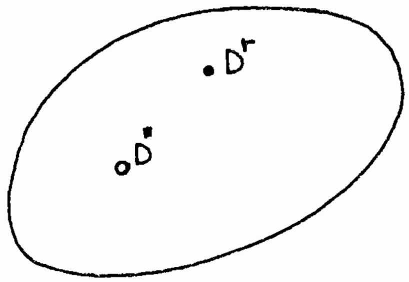

Figure 1

Assume Δ convex and that D^r is a l.f. matrix. By taking out a single matrix D^* the resulting class becomes nonconvex but D^r is still a l.f. matrix. Since it is difficult to know if theorem 0 is valid without assuming convexity, from now on we will deal only with convex classes. The importance of this theorem is that it gives a way to find the l.f. matrix D^r by maximizing the functional $e_{op}(D)$ over the class Δ.

2.4 Models for the class Δ.

Since we are dealing with the density matrix D we need models for classes of matrices. As we said in the beginning we will use the band models (upper and/or lower bounds for the elements of the matrix D). For $D_s(w)$ and $D_n(w)$ it is meaningful to define

$$L_i(w) \leq D_i(w) \leq U_i(w) \qquad i=s,n$$

because $D_s(w)$ and $D_n(w)$ are real functions. For $D_{sn}(w)$ we have to be more careful since it is a complex valued function.

One possible class is defined by the band model for $D_s(w)$ and $D_n(w)$ with no restriction on the cross spectrum $D_{sn}(w)$. This class is convex and theorem 0 can be applied. The solution for this problem is in the next chapter.

We would like also to be able to restrict $D_{sn}(w)$ in some way. We can define a band model for $|D_{sn}(w)|$ which is a real function, so that

$$L(w) \leq |D_{sn}(w)| \leq U(w)$$

Unfortunately this model produces a nonconvex class Δ . There are two ways to overcome this problem . One is by defining :

$$Re(\, D_{sn}(w)\,) \leq 0 \text{ and}$$
$$L(w) \leq |Re(\, D_{sn}(w)\,)| \leq |D_{sn}(w)| \leq U(w)$$

and the resulting class is convex. The second way is to assume that $Re(\, D_{sn}(w)\,)$ can be anything but

$$0 \leq |D_{sn}(w)| \leq U(w)$$

So in the second case we keep only the upper bound. It turns out that we can get the solution to the second class from the solution of the first class just by letting $L(w)=0$.

The assumption that $Re(D_{sn}(w)) \leqslant 0$ seems artificial, but as we will find out in the next section if $Re(D_{sn}(w)) \geqslant 0$ then the l.f. matrix has $D_{sn}(w) =0$. This means that if the $Re(D_{sn}(w))$ is nonnegative it is better to treat the signal and noise as uncorrelated.

In addition to the bounds we also impose power constraints

$$\int_{\infty}^{\infty} D_s(w).dw = 2\pi\sigma_s^2 \qquad \int_{\infty}^{\infty} D_n(w).dw = 2\pi\sigma_n^2$$

where σ_s and σ_n are known numbers. These two power constraints are actually the reason for superiority of robust filters over the bounding filters.

2.5 Maximization of $e_{op}(D)$.

Based on theorem 0 we will try to maximize the $e_{op}(D)$ in order to find the l.f. matrix D^r . From (7) we have that the optimum error is given by:

$$e_{op}(D) = \frac{1}{2\pi}\int_{\infty}^{\infty} \frac{D_s(w).D_n(w) - |D_{sn}(w)|^2}{D_s(w)+D_n(w)+2.Re\left[D_{sn}(w)\right]} dw \qquad (7)$$

For given $D_s(w)$ $D_n(w)$ and $|D_{sn}(w)|$ the worst $Re(D_{sn}(w))$ is $-|D_{sn}(w)|$ because it minimizes the denominator. We have this condition when $D_{sn}(w)=-|D_{sn}(w)|$. Rewriting (7) we have for this case

$$e_{op}(D)=\frac{1}{2\pi}\int_{\infty}^{\infty}\frac{D_s(w).D_n(w)-|D_{sn}(w)|^2}{D_s(w)+D_n(w)-2|D_{sn}(w)|}dw \tag{12}$$

The expression under the integral in (12) for given $D_s(w)$, $D_n(w)$ as a function of $|D_{sn}(w)|$ is :

a. increasing for $0\leq |D_{sn}(w)| \leq \min\{D_s(w), D_n(w)\}$.

b. decreasing for:

$$\min\{D_s(w),D_n(w)\} \leq |D_{sn}(w)| \leq \sqrt{D_s(w)\ D_n(w)}$$

We can easily verify the above statements by taking the derivative with respect to $|D_{sn}(w)|$ assuming $D_s(w)$, $D_n(w)$ constants. From a and b we conclude that for given $D_n(w)$ and $D_s(w)$ the worst $|D_{sn}(w)|$ is the one that is as close as possible to the $\min\{D_s(w),D_n(w)\}$, (but because of the bounds it might not be possible to reach this value). Having the worst $|D_{sn}(w)|$ the worst cross spectral density is given by:

$$D_{sn}(w)= -|D_{sn}(w)|$$

All the above results depend on the assumed band model and especially on the model for $D_{sn}(w)$. If we assume for example that $Re(D_{sn}(w))\geq 0$ then we can easily see that for given $D_s(w)$ and $D_n(w)$ the worst $D_{sn}(w)$ is $D_{sn}(w)=0$ and the optimum error becomes :

$$e_{op}(D)=\frac{1}{2\pi}\int_{\infty}^{\infty}\frac{D_s(w).D_n(w)}{D_s(w)+D_n(w)}\,dw$$

which is nothing else but the optimum error assuming signal and noise uncorrelated. For this case we can use the results in [3] to design the robust filter.

2.6 Uniquness of Robust Filters.

As we will see in the next chapter the l.f.matrix is not unique. Let us assume that D_r^1 and D_r^2 are two l.f. matrices, then

$$e_{op}(D_r^1) = e_{op}(D_r^2) = \max_{D \in \Delta} e_{op}(D) \quad (13)$$

If H_r^1 H_r^2 are the Wiener filters for D_r^1 and D_r^2 (so they are also robust filters) , because of the saddle point relation

$$e_{op}(D_r^1) \geqslant e(D_r^2, H_r^1) \quad (13\alpha)$$

Because of (13), (13α) can be true only as equality :

$$e_{op}(D_r^1) = e(D_r^2, H_r^1) = e_{op}(D_r^2)$$

which means that H_r^1 is a Wiener filter for D_r^2 . If the Wiener filter is unique then also the robust filter is unique. Recalling the expression for the Wiener filter from (5)

$$H_o(w) = \frac{D_s(w) + D_{sn}(w)}{D_s(w)+D_n(w)+2Re[D_{sn}(w)]} \quad (5)$$

the Wiener filter is undefined in the regions where both numerator and denomenator are zero. In these regions it can have any value and is not unique. To overcome this problem we can define the filter to have an arbitrary but always the same value (say 0 or some other constant). Under this restriction the Wiener filter (and so the robust filter) is unique.

CHAPTER 3

Robust Filters for specific Classes.

The various classes will be presented starting from the simple cases and going gradually to more complicated . This way of presentation is prefered because the proofs are easier in the simple cases and they will give us enough backround to understant the more general ones.

3.1 Signal and Noise Given.

A. No bounds on $| D_{sn}(w) |$.

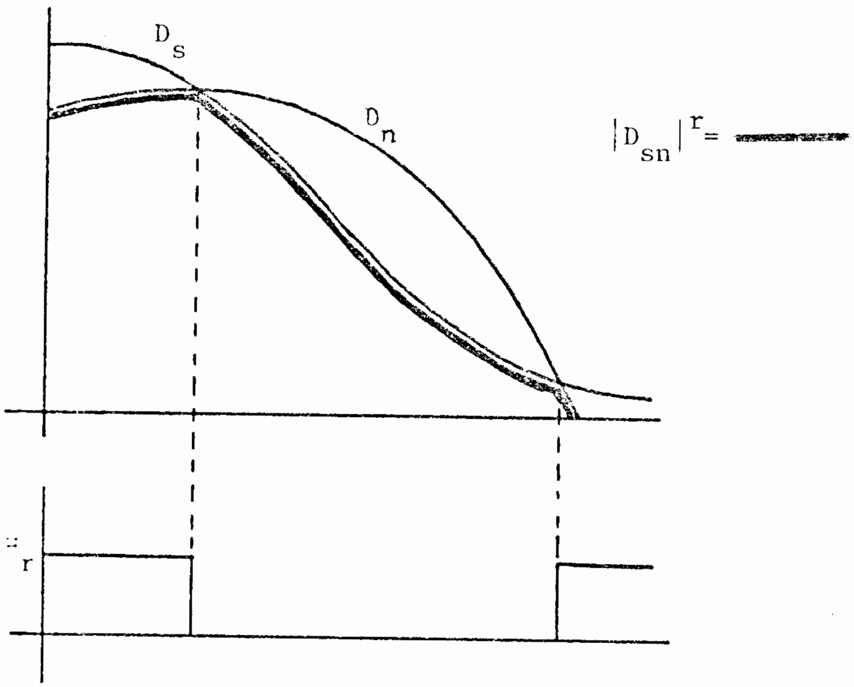

Figure 2.

From the theory in section 2.4 we have that the worst $D_{sn}(w)|$ is $|D_{sn}(w)|^r = \min(D_s(w), D_n(w))$. Also $D^r_{sn} = -|D_{sn}(w)|^r$ and since now we have the l.f. matrix we can design the robust filter. It is given by

$$H_r(w) = \begin{cases} I & \text{when } D_s(w) \geq D_n(w) \\ 0 & \text{otherwise} \end{cases}$$

This filter has the same MSE behavior for any cross correlation. We can verify this from equation (3).

$$e(D,H_r) = \frac{1}{2\pi}\int_{\infty}^{\infty} \{D_s(w) - 2H_r(w)D_{sx}(w) + |H(w)|^2 \cdot D_x(w)\}dw$$

$$= \frac{1}{2\pi}\int_{\infty}^{\infty} \{[I - 2H_r(w) + |H_r(w)|^2] \cdot D_s(w) + |H_r(w)|^2 D_n(w)$$

$$+2\ |H_r(w)|^2 \cdot Re[D_{sn}(w)] - H_r(w)D_{sn}(w)\ \}dw$$

Because $H_r(w)$ is 1 or 0 it is a real function and it is also even as the Fourier transform of a real system . The $Im(D_{sn}(w))$ is an odd function so $\int_{\infty}^{\infty} H_r(w) \cdot Im(D_{sn}(w))\, dw = 0$.

Thus

$$e(D,H_r) = \frac{1}{2\pi}\int_{\infty}^{\infty} \{[I - 2H_r(w) + |H_r(w)|^2] D_s(w) + |H_r(w)|^2 D_n(w) -$$

$$- 2H_r(w)[I - H_r(w)]Re[D_{sn}(w)]\}dw \qquad (I4)$$

But $H_r(w) \cdot [\ H_r(w) - 1\ ] = 0$ and we have

$$e(D,H_r) = \frac{1}{2\pi}\int_{\infty}^{\infty} \{[I - 2H_r(w) + |H_r(w)|^2] D_s(w) + |H_r(w)|^2 D_n(w)\ \}dw$$

which is independent of $D_{sn}(w)$.

B. Bounds on $|D_{sn}(w)|$.

For this case according to section 2.5 the worst $|D_{sn}(w)|^{r}$ is given by

$$|D_{sn}(w)|^r = \text{second-largest}\{ \min[D_s(w), D_n(w)], U(w), L(w) \} \quad (15)$$

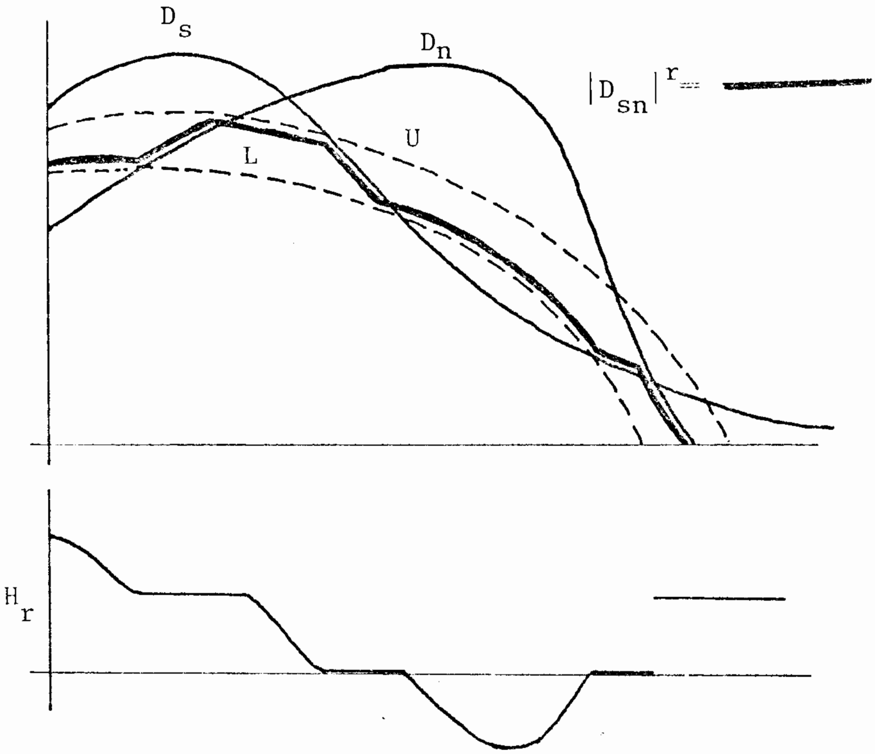

Figure 3.

In figure 3 we can see an illustration of the $|D^r_{sn}(w)|$ and the robust filter. This filter is no longer a simple two level filter ,it can also have negative values and values greater than 1, something that never happens in the uncorrelated signal and noise case.

The model for $D_{sn}(w)$ has to be according to section 2.4 in order the matrix we defined in equation (15) to be a l.f. matrix.

3.2 Band models for $D_s(w), D_n(w)$.

We assume that functions $L_i(w), U_i(w)$ are given such that

$$L_i(w) \leqslant D_i(w) \leqslant U_i(w) \qquad i=s,n$$

We also assume knowledge of the total power of signal and noise

$$\int_{\infty}^{\infty} D_i(w).dw = 2\pi\sigma_i^2 \tag{I6}$$

where σ_i known numbers.

A. No bounds on $|D_{sn}(w)|$.

In section 2.3 we said that for given $D_s(w)$ and $D_n(w)$ if there are no bounds on $|D_{sn}(w)|$ then the worst $|D_{sn}(w)|$ is given by

$$|D_{sn}(w)|^r = \min[D_s(w), D_n(w)]$$

Under this condition the optimum MSE is given by

$$\varepsilon_{op}(D) = \frac{1}{2\pi}\int_{\infty} \frac{D_s . D_n - D_{sn}}{D_s + D_n - 2\ D_{sn}} dw = \frac{1}{2\pi}\int_{\infty}^{\infty} D_s . dw + \frac{1}{2\pi}\int_{\infty}^{\infty} D_n . dw$$

$$= \frac{1}{2\pi}\int_{\infty}^{\infty} \min[D_s, D_n].dw \tag{I7}$$

The l.f. pair $D_s^r(w)$, $D_n^r(w)$ will be the one that will maximize expression (I7) under the power constraints (I6). Obviously if there are no power constraints then (I7) is maximized for $D_s^r(w) = U_s(w)$ and $D_n^r(w) = U_n(w)$. This is nothing but the pair for which the bounding filter is it's optimum filter. Because of the power constraints there are several subcases, but first we will give some definitions

$$A_s(w) = \min\{U_s(w), \max[L_s(w), L_n(w)]\}$$

$$A_n(w) = \min\{U_n(w), \max[L_s(w), L_n(w)]\}$$

$$B_s(w) = \min\{U_s(w), \max[L_s(w), U_n(w)]\}$$

$$B_n(w) = \min\{U_n(w), \max[U_s(w), L_n(w)]\}$$

In figure 4 we illustrate the definition of $A_s(w)$ and $B_s(w)$

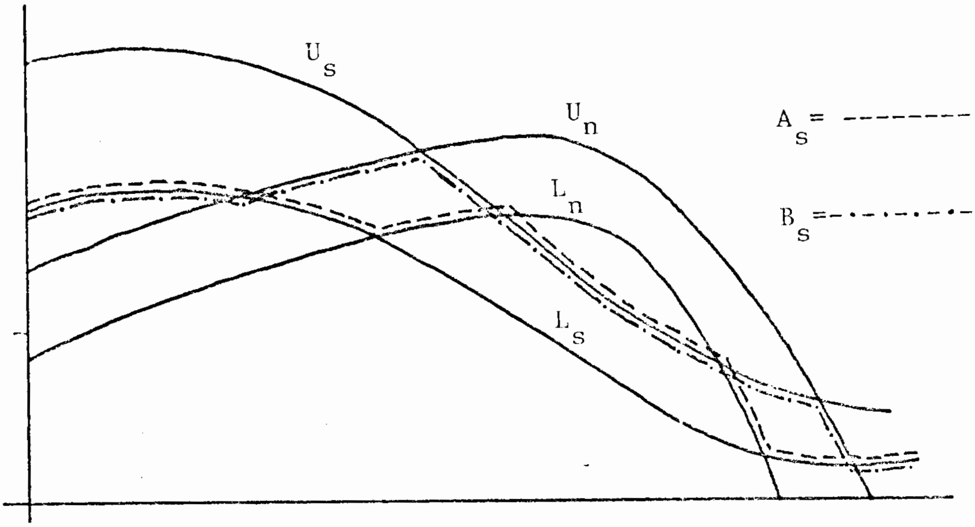

Figure 4.

Now we are going to prove a lemma that will be used in the proofs of the following subcases.

Lemma

Let a,b,c be nonnegative numbers with $a \geq c$, then

$$a - c \geq \min(a,b) - \min(b,c)$$

Proof

$$a - c \geq 0 \geq \min(a,b) - b$$

$$a - c \geq \min(a,b) - c$$

so $\quad a - c \geq \min(a,b) - \min(b,c)$

$$\text{A1.}\quad \int_{\infty}^{\infty} A_s(w)\,.\,dw \geqslant 2\pi\sigma_s^2 \quad , \quad \int_{\infty}^{\infty} A_n(w)\,.\,dw \geqslant 2\pi\sigma_n^2$$

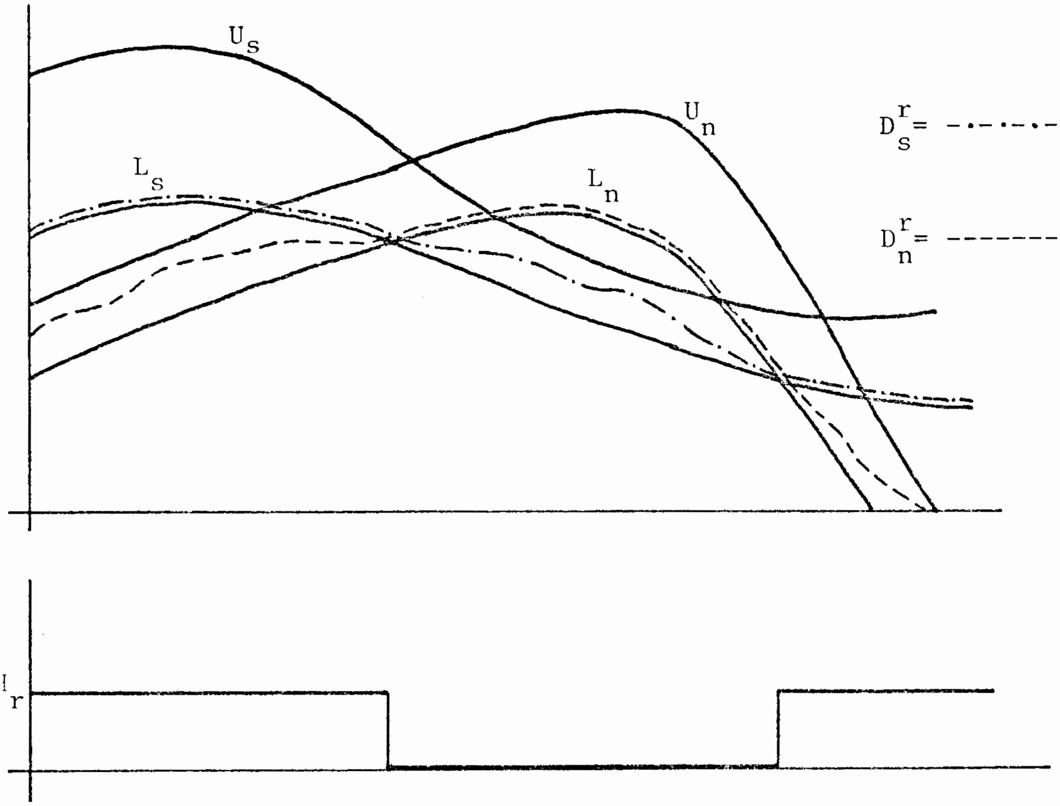

Figure 5

The robust signal and noise densities are given by

$$D_s^r(w) = \begin{cases} L_s(w) & \text{if} \quad A_s(w)=L_s(w) \\ 1_s(w) & \text{otherwise} \end{cases}$$

$$D_n^r(w) = \begin{cases} L_n(w) & \text{if} \quad A_n(w)=L_n(w) \\ 1_n(w) & \text{otherwise} \end{cases}$$

The robust filter is given by

$$H_r(w) = \begin{cases} I & \text{when } L_s(w) \geqslant L_n(w) \\ 0 & \text{otherwise} \end{cases}$$

The MSE optimum error using expression (I7) is

$$e_{op}(D^r) = \sigma_s^2 + \sigma_n^2 - \frac{1}{2\pi}\int_{\infty}^{\infty} \{L_s(w) + L_n(w) - \min[L_s(w), L_n(w)]\} dw$$

where $l_s(w), l_n(w)$ are arbitrary functions with $l_s(w) \leqslant A_s(w)$ and $l_n(w) \leqslant A_n(w)$, but enough for $D_s^r(w)$ and $D_n^r(w)$ to fulfill the power constraints.

<u>Proof.</u>

$$\min[D_s, D_n] = \min[L_s, D_n] + \min[D_s, D_n] - \min[L_s, D_n]$$

because of lemma $\leqslant \min[L_s, D_n] + D_s - L_s$

$$\leqslant D_s - L_s + \min[L_s, L_n] + D_n - L_n$$

and by integrating

$$e_{op}(D) \leq \sigma_s^2 + \sigma_n^2 - \frac{1}{2\pi}\int_{\infty}^{\infty} \{L_s + L_n - \min[L_s, L_n]\} dw$$

A2. $$\int_{\infty}^{\infty} A_s(w).dw \geqslant 2\pi\sigma_s^2 \quad , \quad \int_{\infty}^{\infty} A_n(w).dw < 2\pi\sigma_n^2$$

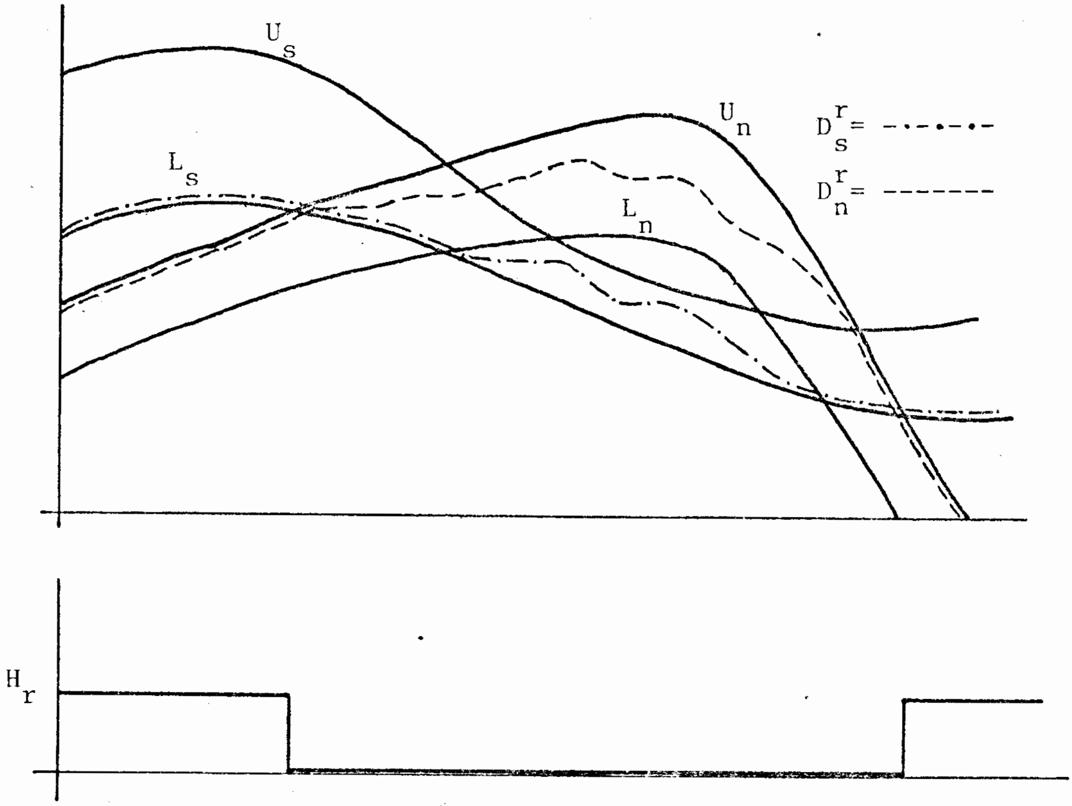

Figure 6.

The robust signal and noise densities are given by

$$D_s^r(w) = \begin{cases} L_s(w) & \text{when} \quad A_s(w)=L_s(w) \\ 1_s(w) & \text{otherwise} \end{cases}$$

$$D_n(w) = \begin{cases} U_n(w) & \text{when} \quad A_n(w)=U_n(w) \\ 1_n(w) & \text{otherwise} \end{cases}$$

The robust filter is given by

$$H_r(w) = \begin{cases} I & \text{when } L_s(w) \geqslant U_n(w) \\ 0 & \text{otherwise} \end{cases}$$

The MSE optimum error is

$$e_{op}(D) = \sigma_s^2 - \frac{1}{2\pi}\int_{\infty}^{\infty}\{L_s(w) - \min[L_s(w), U_n(w)]\}dw$$

where $l_s(w), l_n(w)$ arbitrary functions with $l_s(w) \leqslant A_s(w)$ and $l_n(w) \geqslant A_n(w)$, but enough again to fulfill the power constraints.

<u>Proof.</u>

$$\min[D_s, D_n] \leqslant \min[D_s, U_n] = \min[L_s, U_n] + \min[D_s, U_n] - \min[L_s, U_n]$$

using lemma $\leqslant \min[L_s, U_n] + D_s - L_s$

by integrating

$$e_{op}(D) \leqslant \sigma_s^2 - \frac{1}{2\pi}\int_{\infty}^{\infty}\{L_s - \min[L_s, U_n]\}dw$$

A3. $\left| \int_{\infty}^{\infty} A_s(w).dw \leq 2\pi\sigma_s^2 \quad , \quad \int_{\infty}^{\infty} A_n(w).dw \geq 2\pi\sigma_n^2 \right|$

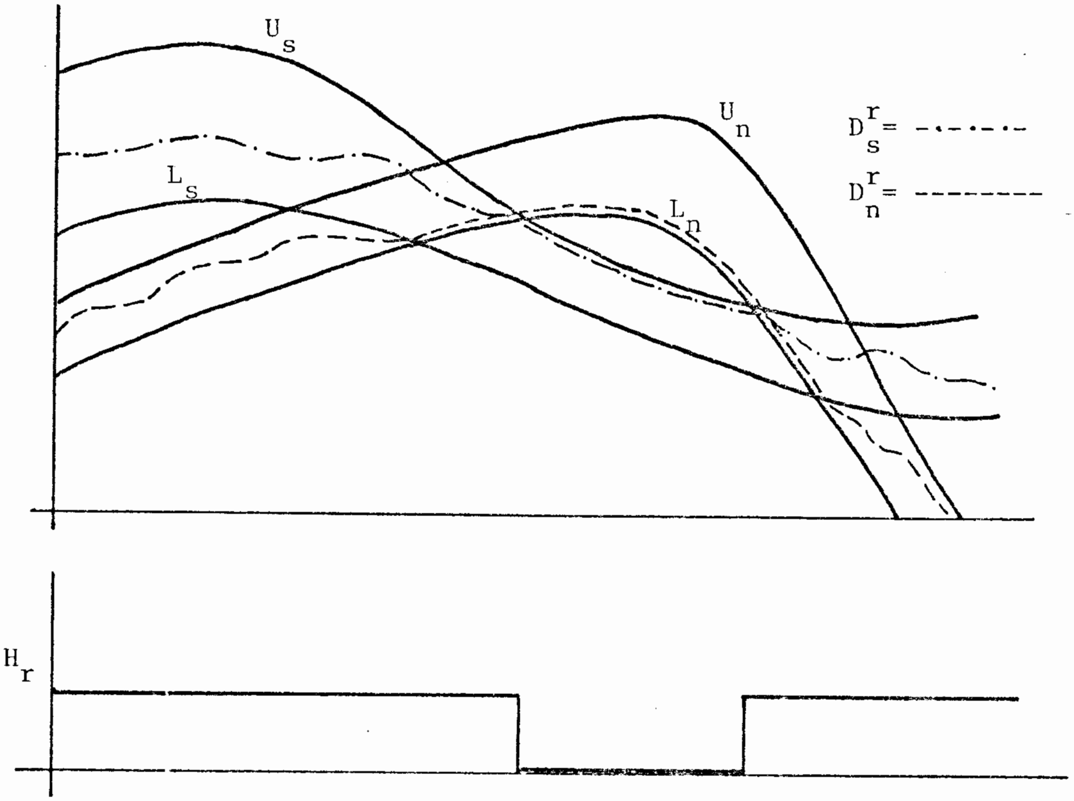

Figure 7.

The robust signal and noise densities are given by

$$D_s(w) = \begin{cases} U_s(w) & \text{when} \quad U_s(w)=A_s(w) \\ 1_s(w) & \text{otherwise} \end{cases}$$

$$D_n(w) = \begin{cases} L_n(w) & \text{when} \quad L_n(w)=A_n(w) \\ 1_s(w) & \text{otherwise} \end{cases}$$

The robust filter is given by

$$H_r(w) = \begin{cases} I & \text{when } U_s(w) \geq L_n(w) \\ 0 & \text{otherwise} \end{cases}$$

The MSE error is

$$e_{op}(w) = \sigma_n^2 - \frac{1}{2\pi}\int_{\infty}^{\infty}\{L_n(w)-\min[L_n(w),U_s(w)]\}dw$$

where $1_n(w) \leq A_n(w)$ and $1_s(w) \geq A_s(w)$ and enough for the power constraints.

Proof.

$$\min[D_s,D_n] \leq \min[U_s,D_n]=\min[U_s,L_n]-\min[U_s,L_n]+\min[U_s,L_n]$$

using lemma $\leq \min[U_s,L_n] + D_n-L_n$

by integrating

$$e_{op}(D) \leq \sigma_n^2 - \frac{1}{2\pi}\int_{\infty}^{\infty}\{L_n-\min[U_s,L_n]\}dw$$

A4. $\int_{\infty}^{\infty} B_s(w).dw \geq 2\pi\sigma_s^2 > \int_{\infty}^{\infty} A_s(w).dw, \quad \int_{\infty}^{\infty} B_n(w).dw \geq 2\pi\sigma_n^2 > \int_{\infty}^{\infty} A_n(w).dw$

$$2\pi\sigma_n^2 - \int_{\infty}^{\infty} A_n(w).dw \geq 2\pi\sigma_s^2 - \int_{\infty}^{\infty} A_s(w).dw$$

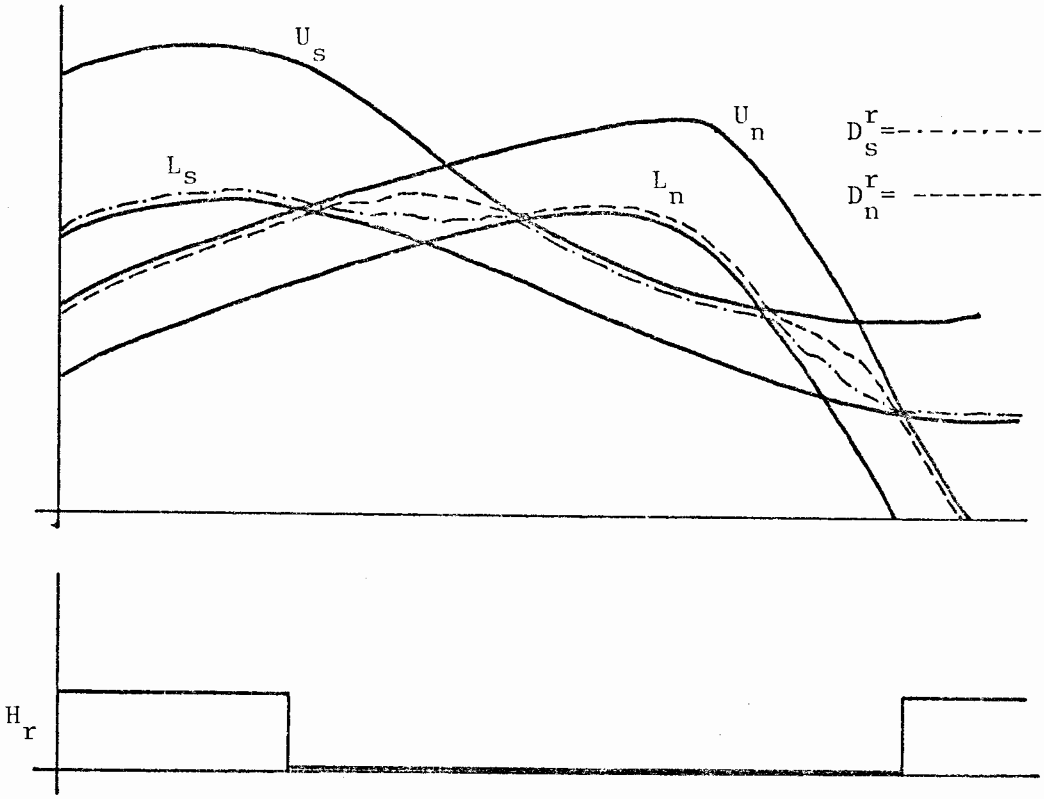

Figure 8.

The robust signal and noise spactrum pair is given by

$$D_s^r(w) = \begin{cases} A_s(w) & \text{when} \quad A_s(w) = B_s(w) \\ l_s(w) & \text{otherwise} \end{cases}$$

$$D_n^r(w) = \begin{cases} A_n(w) & \text{when} \quad A_n(w) = B_n(w) \\ l_n(w) & \text{otherwise} \end{cases}$$

The robust filter is given by

$$H_r(w) = \begin{cases} I & \text{when} \quad L_s(w) \geqslant U_n(w) \\ 0 & \text{otherwise} \end{cases}$$

The optimum MSE is

$$e_{op}(D^r) = \sigma_s^2 - \frac{1}{2\pi}\int_{\infty}^{\infty} \{L_s(w) - \min[U_n(w), L_s(w)]\} dw$$

Where $A_s(w) \leqslant l_s(w) \leqslant B_s(w)$ and $D_s^r(w) \leqslant l_n(w) \leqslant B_n(w)$

<u>Proof.</u>

$$\min[D_s, D_n] \leqslant \min[D_s, U_n] = \min[U_n, L_s] + \min[D_s, U_n] - \min[U_n, L_s]$$

using the lemma $\qquad \leqslant \min[U_n, L_s] + D_s - L_s$

by integrating

$$e_{op}(D) \leqslant \sigma_s^2 - \frac{1}{2\pi}\int_{\infty}^{\infty} \{L_s - \min[L_s, U_n]\} dw$$

A5. $\int_{\infty}^{\infty} B_s(w).dw \geq 2\pi\sigma_s^2 > \int_{\infty}^{\infty} A_s(w).dw \ , \ \int_{\infty}^{\infty} B_n(w).dw \geq 2\pi\sigma_n^2 > \int_{\infty}^{\infty} A_n(w).dw$

$$2\pi\sigma_s^2 - \int_{\infty}^{\infty} A_s(w).dw > 2\pi\sigma_n^2 - \int_{\infty}^{\infty} A_n(w).dw$$

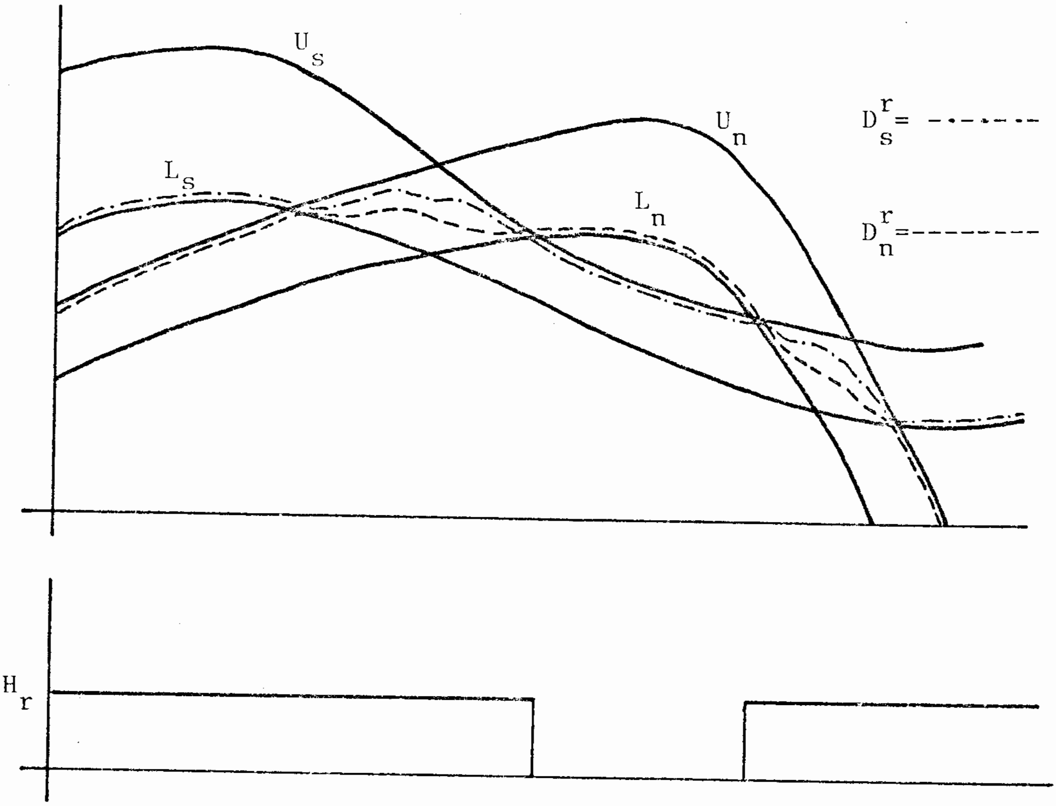

Figure 9.

The robust signal and noise densities pair is given by

$$D_s^r(w) = \begin{cases} A_s(w) & \text{when} \quad A_s(w) = B_s(w) \\ l_s(w) & \text{otherwise} \end{cases}$$

$$D_n^r(w) = \begin{cases} A_n(w) & \text{when} \quad A_n(w) = B_n(w) \\ l_n(w) & \text{otherwise} \end{cases}$$

The robust filter is given by

$$H_r(w) = \begin{cases} I & U_s(w) \geq L_n(w) \\ 0 & \text{otherwise} \end{cases}$$

The optimum MSE is

$$e_{op}(D^r) = \sigma_n^2 - \frac{1}{2\pi}\int_{\infty}^{\infty} \{L_n(w) - \min|U_s(w), L_n(w)|\} dw$$

where $A_n(w) \leqslant 1_n(w) \leqslant B_n(w)$ and $D_n^r(w) \leqslant 1_s(w) \leqslant B_s(w)$

Proof

$$\min[D_s, D_n] \leqslant \min[D_n, U_s] = \min[L_n, U_s] + \min[D_n, U_s] - \min[L_n, U_s]$$

using lemma $\leqslant \min[L_n, U_s] + D_n - L_n$

and by integrating

$$e_{op}(D) \leqslant \sigma_n^2 - \frac{1}{2\pi}\int_{\infty}^{\infty} \{L_n - \min[L_n, U_s]\} . dw$$

$$\text{A6.}\quad \int_{\infty}^{\infty} B_s(w).dw \geqslant 2\pi\sigma_s^2 > \int_{\infty}^{\infty} A_s(w).dw \ , \quad 2\pi\sigma_n^2 > \int_{\infty}^{\infty} B_n(w).dw$$

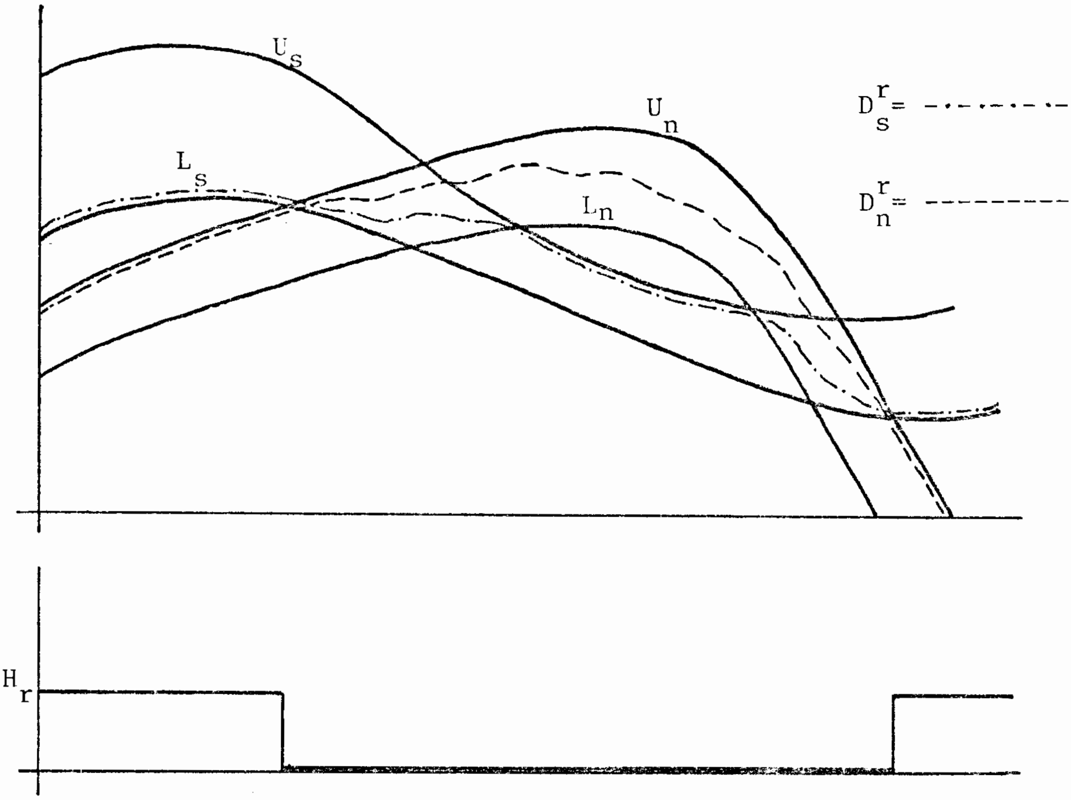

Figure 10.

The robust signal and noise densities pair is given by

$$D_s^r(w) = \begin{cases} A_s(w) & \text{when} \quad A_s(w) = B_s(w) \\ \\ 1_s(w) & \text{otherwise} \end{cases}$$

$$D_n^r(w) = \begin{cases} U_n(w) & \text{when} \quad U_n(w) = A_n(w) \\ \\ 1_n(w) & \text{otherwise} \end{cases}$$

The robust filter is

$$H_r(w) = \begin{cases} I & \text{when } L_s(w) \geq U_n(w) \\ 0 & \text{otherwise} \end{cases}$$

The optimum MSE is given

$$e_{op}(D^r) = \sigma_s^2 - \frac{1}{2\pi} \int_{\infty}^{\infty} \{L_s - \min[U_n, L_s]\} dw$$

Where $A_s(w) \leq l_s(w) \leq B_s(w)$ and $D_s^r(w) \leq l_n(w)$

<u>Proof</u>

$$\min[D_s, D_n] \leqslant \min[D_s, U_n] = \min[L_s, U_n] + \min[D_s, U_n] - \min[L_s, U_n]$$

$$\leqslant \min[L_s, U_n] + D_s - L_s$$

and by integrating

$$e_{op}(D) \leqslant \sigma_s^2 - \frac{1}{2\pi} \int_{\infty}^{\infty} \{L_s - \min[U_n, L_s]\} dw$$

A7. $\int_{\infty}^{\infty} B_n(w).dw \geqslant 2\pi\sigma_n^2 > \int_{\infty}^{\infty} A_n(w).dw, \quad 2\pi\sigma_s^2 > \int_{\infty}^{\infty} B_s(w).dw$

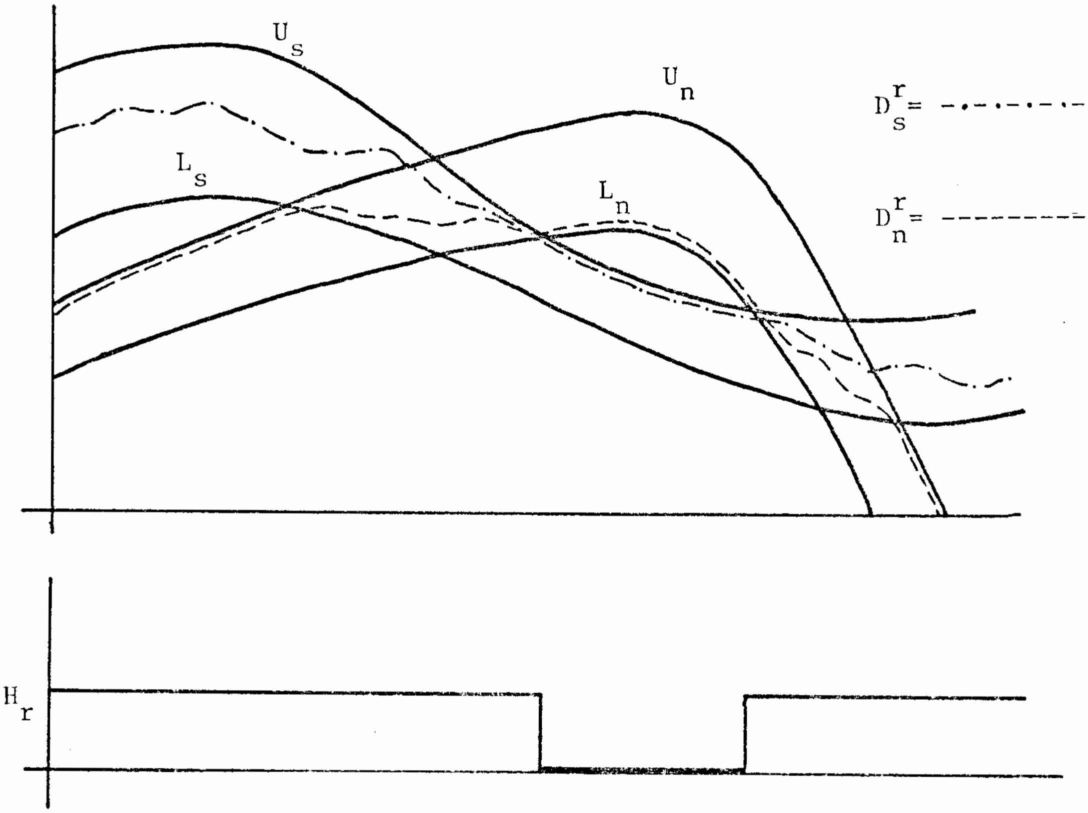

Figure 11.

The robust signal and noise densities pair is

$$D_s^r(w) = \begin{cases} U_s(w) & \text{when} \quad U_s(w) = B_s(w) \\ 1_s(w) & \text{otherwise} \end{cases}$$

$$D_n^r(w) = \begin{cases} A_n(w) & \text{when} \quad A_n(w) = B_n(w) \\ 1_n(w) & \text{otherwise} \end{cases}$$

The robust filter is

$$H_r(w) = \begin{cases} 1 & \text{when} \quad U_s(w) \geqslant L_n(w) \\ 0 & \text{otherwise} \end{cases}$$

The optimum MSE is given by

$$e_{op}(D^r) = \sigma_s^2 - \frac{1}{2\pi} \int \{L_n(w) - \min[U_n(w), L_s(w)]\}dw$$

Where $A_n(w) \leqslant 1_n(w) \leqslant B_n(w)$ and $D_n^r(w) \leqslant 1_s(w)$

<u>Proof</u>

$$\min[D_s, D_n] \leqslant \min[D_s, U_n] = \min[L_s, U_n] + \min[D_s, U_n] - \min[L_s, U_n]$$

using lemma $\leqslant \min[L_s, U_n] + D_s - L_s$

and integrating

$$e_{op}(D) \leqslant \sigma_s^2 - \frac{1}{2\pi} \int_{\infty}^{\infty} \{L_s - \min[L_s, U_n]\}dw$$

A8. $$2\pi\sigma_s^2 > \int_{\infty}^{\infty} B_s(w).dw \;, \quad 2\pi\sigma_n^2 > \int_{\infty}^{\infty} B_n(w).dw$$

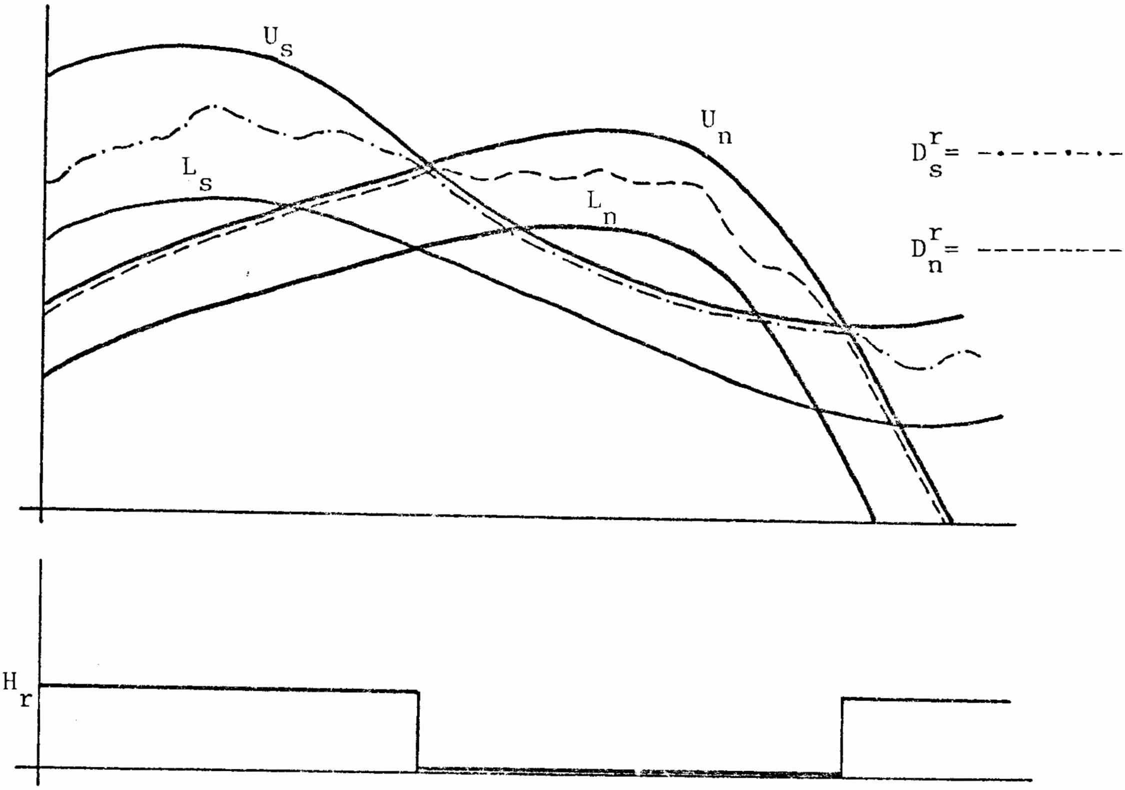

Figure 12.

The robust signal and noise densities pair is

$$D_s^r(w) = \begin{cases} U_s(w) & \text{when } U_s(w) = B_s(w) \\ l_s(w) & \text{otherwise} \end{cases}$$

$$D_n^r(w) = \begin{cases} U_n(w) & \text{when } U_n(w) = B_n(w) \\ l_n(w) & \text{otherwise} \end{cases}$$

The robust filter is

$$H_r(w) = \begin{cases} 1 & \text{when} \quad U_s(w) \geq U_n(w) \\ 0 & \text{otherwise} \end{cases}$$

The optimum MSE is given by

$$e_{op}(D^r) = \frac{1}{2\pi}\int_{\infty}^{\infty} \text{nin}\left[U_s(w), U_n(w)\right] dw$$

Where $B_s(w) \leq 1_s(w)$ and $B_n(w) \leq 1_n(w)$.

<u>Proof</u>

$$\min\left[D_s(w), D_n(w)\right] \leq \min\left[U_s(w), U_n(w)\right]$$

and by integrating

$$e_{op}(D) \leq \frac{1}{2\pi}\int_{\infty}^{\infty} \min\left[U_s(w), U_n(w)\right] dw$$

B. Upper bound on $|D_{sn}(w)|$.

Let us assume that a nonnegative function R(w) is given such that

$$0 \leq |D_{sn}(w)| \leq R(w) \leq \min [L_s(w), L_n(w)]$$

Here we have the special case where the upper bound of $|D_{sn}(w)|$ is less than the two lower bounds of signal and noise.

Because of the maximization problem $|D_{sn}(w)|$. has to be as close as possible to $\min[D_s(w), D_n(w)]$. The closest it can be is when $|D_{sn}(w)| = R(w)$ Since we have specified the worst cross densities we have to specify now the robust signal and noise pair. The optimum MSE is given by

$$e_{op}(D) = \frac{1}{2\pi}\int_{\infty}^{\infty} \frac{D_s(w).D_n(w)-[R(w)]^2}{D_s(w)+D_n(w)-2R(w)} dw$$

or after some manipulations

$$e_{op}(D) = \frac{1}{2\pi}\int_{\infty}^{\infty} R(w).dw + \frac{1}{2\pi}\int_{\infty}^{\infty} \frac{S(w).N(w)}{S(w) + N(w)} \qquad (17)$$

Where $S(w)=D_s(w)-R(w) \geq 0$ and $N(w)=D_n(w)-R(w) \geq 0$

To maximize expression (17) it is enough to maximize the second integral because the first is a constant. But this term is the expression for the min MSE for uncorrelated signal S(w) and noise N(w) with bounds

$$S_L(w)=L_s(w)-R(w) \leq S(w) \leq U_s(w)-R(w)=S_U(w)$$

$$N_L(w)=L_n(w)-R(w) \leq N(w) \leq U_n(w)-R(w)=N_U(w)$$

and power constraints

$$\int_{\infty}^{\infty} S(w)\,dw = 2\pi\sigma_s^2 - \int_{\infty}^{\infty} R(w)\,dw$$

$$\int_{\infty}^{\infty} N(w)\,dw = 2\pi\sigma_n^2 - \int_{\infty}^{\infty} R(w)\,dw$$

This is exactly the problem solved in [3] and the solution is given by the following subcases

$$S_r(w)=\begin{cases} k_s N_L(w) & \text{when } S_L(w) \leq k_s N_L(w) \leq S_U(w) \\ S_L(w) & k_s N_L(w) < S_L(w) \\ S_U(w) & k_s N_L(w) > S_U(w) \end{cases}$$

$$N_r(w)\begin{cases} \frac{1}{k_n} S_L(w) & \text{when } N_L(w) \leq \frac{1}{k_n} S_L(w) \leq N_U(w) \\ N_L(w) & \frac{1}{k_n} S_L(w) > N_L(w) \\ N_U(w) & N_U(w) < \frac{1}{k_n} S_L(w) \end{cases}$$

if $k_s < k_n$ exists satisfing the power constraints;

if there is no such $k_s < k_n$ then

$$S_r(w)=\begin{cases} kN_L(w)+Se(w) & \text{when } S_L(w) \leq kN_L(w) \leq S_U(w) \\ S_U(w) & S_U(w) < kN_L(w) \\ S_L(w)+Se(w) & N_L(w) \leq \frac{1}{k} S_L(w) \leq N_U(w) \\ S_L(w) & N_U(w) < \frac{1}{k} S_L(w) \end{cases}$$

$$N_r(w)=\begin{cases} \frac{1}{k} S_L(w)+Ne(w) & \text{when } N_L(w) \leq \frac{1}{k} S_L(w) \leq N_U(w) \\ N_U(w) & N_U(w) < \frac{1}{k} S_L(w) \\ N_L(w)+Ne(w) & S_L(w) \leq kN_L(w) \leq S_U(w) \\ N_L(w) & S_U(w) < kN_L(w) \end{cases}$$

where k enough for the power constraints and $Se(w) = k\,Ne(w)$ but otherwise arbitrary nonnegative functions.

If neither of the above true then

$$S_r(w) = \begin{cases} 1_s N_U(w) & S_L(w) < 1_s N_U(w) < S_U(w) \\ S_L(w) & 1_s N_U(w) < S_L(w) \\ S_U(w) & 1_s N_U(w) > S_U(w) \end{cases}$$

$$N_r(w) = \begin{cases} S_U(w) & N_L(w) < S_U(w) < N_U(w) \\ N_L(w) & \frac{1}{1_n} S_U(w) < N_L(w) \\ N_U(w) & \frac{1}{1_n} S_U(w) > N_U(w) \end{cases}$$

with $1_s > 1_n$ and such that the power constraints are satisfied. If there is no solution for 1_s then $S_r(w)=S_U(w)$ when $N_U(w) > 0$ and arbitrary otherwise and if 1_n has no solution then $N_r(w)=N_U(w)$ when $S_U(w) > 0$ and arbitrary otherwise.

The proof can be found in [3].

3.3 Given $D_s(w)$, Bounds on $D_n(w)$ and $|D_{sn}(w)|$.

For this case we assume that functions $D_s(w)$ $L_n(w), U_n(w), L(w), U(w)$ are given such that

$$L_n(w) \leqslant D_n(w) \leqslant U_n(w)$$

$$L(w) \leqslant |D_{sn}(w)| \leqslant U(w)$$

with $$L(w) \leqslant \sqrt{D_s(w) \cdot L_n(w)}$$

The model for $D_{sn}(w)$ has to be according to section 2.4 .

When $L(w) \neq 0$ then

$$Re(D_{sn}(w)) \leqslant 0 \text{ and } |Re(D_{sn}(w))| \geqslant L(w)$$

When $L(w) = 0$ then $D_{sn}(w)$ can be anything. Also power constraint for the noise process

$$\int_{\infty}^{\infty} D_n(w).dw = 2\pi\sigma_n^2$$

where σ_n is a given number. Because of the power constraint several subcases arise.

Let us define first

$$F(w) = k\,D_s(w) + (1-k)\,U(w)$$

$$G(w) = (k+3)\,L(w) - (k+2)\,D_s(w)$$

$$E(w) = k\,D_s(w) + (1-k)\,L(w)$$

when $D_n^r(w)$ is given by one of the following subcases A,B,C or D.

A.

A1. $D_s(w) \geq L(w)$.

a_1.	$E(w)$	when $L_n(w) \leq E(w) \leq U_n(w)$
a_2.	$L_n(w)$	$E(w) < L_n(w)$
a_3.	$U_n(w)$	$E(w) > U_n(w)$

A2. $L(w) > D_s(w)$.

a_4.	$G(w)$	when $L(w) \leq G(w) \leq U(w)$
a_5.	$L_n(w)$	$L_n(w) < G(w)$
a_6.	$U_n(w)$	$U_n(w) > G(w)$

Where $0 \geq k \geq -1$ and such that the power constraint is satisfied.

B.

B1. $D_s \geq L(w)$

$U_n(w)$ when $L > U_n$

$l(w)$ when $\max\{L, L_n\} \leq \min\{U, U_n, D_s$

L_n when $L_n \geq \min\{D_s, U\}$

B1. $D_s(w) \geq U(w)$.

b1. $l(w)$

B2. $U(w) > D_s(w) \geq L(w)$.

b2. $l(w)$ when $L_n(w) \leq D_s(w) \leq U_n(w)$

b3. $L_n(w)$ $D_s(w) < L_n(w)$

b4. $U_n(w)$ $D_s(w) > U_n(w)$

B3. $L(w) > D_s(w)$.

b5. $3 \cdot L(w) - 2 \cdot D_s(w)$ when $L_n(w) \leq 3 \cdot L(w) - 2 \cdot D_s(w) \leq U_n(w)$

b6. $L_n(w)$ $L_n(w) > 3 \cdot L(w) - 2 \cdot D_s(w)$

b7. $U_n(w)$ $U_n(w) < 3 \cdot L(w) - 2 \cdot D_s(w)$

where $l(w)$ is an arbitrary function such that

$$\max\{L(w), L_n(w)\} \leq l(w) \leq \min\{U(w), U_n(w), D_s(w)\}$$

and enough to satisfy the power constraint.

C.

C1. $D_s(w) \geq U(w)$.

c1. $F(w)$ when $L_n(w) \leq F(w) \leq U_n(w)$

c2. $L_n(w)$ $L_n(w) > F(w)$

c3. $U_n(w)$ $U_n(w) < F(w)$

C2. $U(w) > D_s(w) \geq L(w)$.

c4. $D_s(w)$ when $L_n(w) \leq D_s(w) \leq U_n(w)$

c5. $L_n(w)$ $L_n(w) < D_s(w)$

c6. $U_n(w)$ $U_n(w) < D_s(w)$

C3. $L(w) > D_s(w)$.

c7. $G(w)$ when $L_n(w) \leq G(w) \leq U_n(w)$

c8. $L_n(w)$ $L_n(w) < G(w)$

c9. $U_n(w)$ $U_n(w) > G(w)$

with k a positive constant and selected in order for $D_n^{r}(w)$ to fulfill the power constraint.

D.

D1. $\min\{U_n(w), U(w)\} \geq D_s(w) \geq L(w)$.

d1. $l(w)$

D2. Otherwise.

d2. $U_n(w)$

where for $l(w)$ we have $U_n(w) \geq l(w) \geq \max[D_s(w), L_n(w)]$ and arbitrary otherwise but enough to satisfy the power constraint.

As we go from subcase A to subcase D the integral $\int D_n^r(w)dw$ increases continuously from $\int L_n(w)dw$ to $\int U_n(w)dw$. Because $L_n(w) \leq D_n(w) \leq U_n(w)$ by integrating

$$\int_{\infty}^{\infty} L_n(w)\,dw \leq 2\pi\sigma_n^2 \leq \int_{\infty}^{\infty} U_n(w)\,dw$$

Thus for given power one of the subcases is the solution to the problem.

Proof

To prove that the expressions given above are the l.f $D_n(w)$ we can show that

$$e_{op}(D^r) \geq e(D,H_r)$$

for any matrix D. The $D^r_{sn}(w)$ is a real and nonpositive function and because of (5) $H_r(w)$ is also real. From (14) we have

$$e(D,H_r)=\frac{1}{2\pi}\int_{\infty}^{\infty}\{[1-2H_r+|H_r|^2]D_s+|H_r|^2 D_n-2H_r(1-H_r)Re[D_{sn}]\}dw$$

and since $H_r(w)$ is real

$$e(D,H_r)=\frac{1}{2\pi}\int_{\infty}^{\infty}\{(1-H_r)^2 D_s+(H_r)^2 D_n-2H_r(1-H_r)Re[D_{sn}]\}dw \qquad (18)$$

$$e(D^r,H_r)-e(D,H_r)=\frac{1}{2\pi}\int_{\infty}^{\infty}H_r^2(D_n^r-D_n)dw +$$

$$+\frac{1}{\pi}\int_{\infty}^{\infty}H_r(1-H_r)[Re(D_{sn})-Re(D^r_{sn})] dw$$

It is enough to show that each integral above is nonnegative. The $|D_{sn}(w)|^r$ because of section 2.5 is defined

$$|D_{sn}(w)|^r=\text{second-largest}\{\min[D_s(w),D_n^r(w)],U(w),L(w)\} \qquad (19)$$

and it can be $D_s(w)$, $D_n^r(w)$, $U(w)$ or $L(w)$. Also the robust filter is

$$H_r(w)=\frac{D_s-|D^r_{sn}|}{D_s-|D^r_{sn}|+D^r_n-|D^r_{sn}|}$$

When $|D^r_{sn}(w)|=D_s(w)$ as we can see $H_r(w)=0$. The integral

$$\int_{\infty}^{\infty}H_r(1-H_r)[Re(D_{sn})-Re(D^r_{sn})]dw$$

will be also zero. When $|D_{sn}(w)| = D_n^r(w)$ then $H_r(w)=1$ and the above integral is zero again. When $|D_{sn}(w)|^r = U(w)$ then $H_r(w) = \dfrac{D_s - U}{D_s - U + D_n^r - U}$ and the integral above becames

$$\int \frac{(D_s - U)\cdot(D_n^r - U)}{(D_s - U + D_n^r - U)} \left[Re(D_{sn}) - Re(D_{sn}^r)\right] dw$$

But $|D_{sn}(w)| \leq U(w)$ so $U(w) + Re(\ D_{sn}(w)\ ) \geq 0$. Also in order to have $|D_{sn}(w)|^r = U(w)$ we need $\min[D_s(w), D_n^r(w)] \geq U(w)$. So all the terms in the integral are nonnegative. When $|D_{sn}(w)|^r = L(w)$ we must have

$$\min[D_s(w), D_s^r(w)] \leq L(w) \leq \max[D_s(w), D_n(w)]$$

This means that $\{D_s(w) - L(w)\}\cdot\{D_n(w) - L(w)\} \leq 0$. The integral for this case is

$$\int \frac{(D_s - L)\cdot(D_n^r - L)}{(D_s - L + D_n^r - L)} \left[Re(D_{sn}) - Re(D_{sn}^r)\right] dw$$

As we said in section 2.4 when we use $L(w) \neq 0$ we must also have that $|Re(\ D_{sn}(w)\ )| \geq L(w)$. Again all the terms in the integral are nonnegative. For every possible case we proved that the integral is a nonnegative number. We must also prove that the integral

$$I = \int (H_r)^2 \left[D_n^r - D_n\right] \cdot dw$$

is nonnegative. We will show this for every individual subcase.

Subcase A.

A1. $D_s(w) \geq L(w)$.

Because $k \leq 0$ we have

$$E(w)=kD_s(w)+(1-k)L(w) \leq L(w)$$

case a_1. Here $D_n^r(w)=E(w) \leq L(w)$ and because also $D_s(w) > L(w)$ from (19) we have that $|D_{sn}(w)|^r=L(w)$ The filter is

$$H_r(w)=\frac{D_s-L}{D_s-L+D_n^r-L} = \frac{1}{1+k}$$

and the integral for the set of w where case a_1 is true is

$$I_{a_1} = \int_{a_1} (H_r)^2 . (D_n^r-D_n)\,dw = \frac{1}{(1+k)^2}\int_{a_1} (D_n^r-D_n)\,dw$$

case a_2.

Here $D_n^r(w)=L_n(w)$ the $|D_{sn}(w)|^r$ is always given from (19). We will show that $H_r(w)=\frac{1}{(1+k)}$. When $\min[D_s,D_n^r] \geq U$ then $|D_{sn}(w)|^r=U(w)$ and because $-1 < k \leq 0$ the filter is

$$H_r= \frac{D_s-U}{D_s-U+D_n^r-U} \leq 1 \leq \frac{1}{(1+k)}$$

When $|D_{sn}(w)|^r=\min[D_s(w),D_n^r(w)]$ then again

$$H_r= \frac{D_s- D_{sn}}{D_s- D_{sn}+D_n- D_{sn}} \leq 1\leq \frac{1}{(1+k)}$$

When $L_n(w) < L(w)$ because $D_s(w) \geq L(w)$ from (19) we have that $|D_{sn}(w)|^r = L(w)$ and the filter is

$$H_r = \frac{D_s - L}{D_s - L + D_n^r - L}$$

Now by replacing $D_n^r(w)$ by $E(w) \leqslant D_n^r(w)$ the filter becames

$$H_r = \frac{D_s - L}{D_s - L + D_n^r - L} \leqslant \frac{1}{(1+k)}$$

Thus for every possible case we proved that $H_r(w) \geqslant \frac{1}{(1+k)}$.

Because $D_n^r(w) = L_n(w)$ we have $D_n^r(w) - D_n(w) \leqslant 0$ and the integral where case a_2 is true is

$$I_a = \int_{a_2} (H_r)^2 (D_n^r - D_n)\, dw \geqslant \frac{1}{(1+k)^2} \int_{a_2} (D_n^r - D_n)\, dw$$

case a $_3$.

Here $D_n^r(w) = U_n(w) < E(w) \leqslant L(w)$ and because $D_s(w) \geqslant L(w)$ from (19) we have $|D_{sn}(w)|^r = L(w)$. The filter is

$$H_r = \frac{D_s - L}{D_s - L + D_n^r - L}$$

By replacing $D_n^r(w)$ with $E(w) < D_n^r(w)$ the filter is

$$H_r = \frac{D_s - L}{D_s - L + D_n^r - L} \leqslant \frac{1}{(1+k)}$$

Because $D_n^r(w) = U_n(w)$ we will have $D_n^r(w) - D_n(w) \geqslant 0$ and the integral is

$$I_a = \int_{a_3} (H_r)^2 (D_n^r - D_n)\, dw \geqslant \frac{1}{(1+k)^2} \int_{a_3} (D_n^r - D_n)\, dw$$

A2. $D_s(w) < L(w)$.

Because $D_s(w) < L(w)$ from (19) we have that always $|D^r_{sn}(w)| = L(w)$.

case a_4. Here $D^r_n(w)=G(w)$ so the filter becames

$$H_r = \frac{D_s - L}{D_s - L + G - L} = - \frac{1}{(1+k)}$$

The integral for this case is

$$I_a = \int_{a_4} (H_r)^2 (D^r_n - D_n)\,dw = \frac{1}{(1+k)^2} \int_{a_4} (D^r_n - D_n)\,dw$$

case a_5. For this case $D^r_n(w)=L_n(w) > G(w)$ and the filter is

$$H_r = \frac{D_s - L}{D_s - L + D^r_n - L}$$

By replacing $D^r_n(w)$ with $G(w) < D^r_n(w)$ the filter becames

$$(H_r)^2 = \left[\frac{D_s - L}{D_s - L + D^r_n - L}\right]^2 \leq \frac{1}{(1+k)^2}$$

Because $D^r_n(w)=L_n(w)$ we have $D^r_n(w) - D_n(w) \leq 0$ so the integral for this case is

$$I_a = \int_{a_5} (H_r)^2 (D^r_n - D_n)\,dw \geq \frac{1}{(1+k)^2} \int_{a_5} (D^r_n - D_n)\,dw$$

case a_6: Here $D_n(w) = U_n(w) < G(w)$ and the filter is

$$H_r = \frac{D_s - L}{D_s - L + U_n - L}$$

By replacing $D_n^r(w)$ with $G(w) < D_n^r(w)$ the filter is

$$[H_r]^2 = [\frac{D_s-L}{D_s-L+D_n^r-L}]^2 > [\frac{D_s-L}{D_s-L+G-L}]^2 = \frac{1}{(1+k)^2}$$

Because $D_n^r(w)=U_n(w)$ we have $D_n^r(w)-D_n(w) \geq 0$, and the integral is

$$I_{a6} = \int_{a6} (H_r)^2 \cdot (D_n^r-D_n)dw \geq \frac{1}{(1+k)^2}\int_{a6} (D_n^r-D_n)\, dw$$

For every case a_i we proved that

$$I_{a_i} \geq \frac{1}{(1+k)^2}\int_{a_i} (D_n^r-D_n)dw$$

By adding all the different cases

$$I = \Sigma I_{a_i} \geq \frac{1}{(1+k)^2}\int_{a_i} (D_n^r-D_n)\cdot dw \quad =0$$

<u>Subcase B.</u>

<u>B1. $D_s(w) \geq U(w)$</u>

<u>case b_1.</u> Here $D_n^r(w)=l(w)$ an arbitrary function with

$\max\{L(w), L_n(w)\} \leq l(w) < \min\{U(w), U_n(w), D_s(w)\}$

Because $D_s(w) \geq U(w)$ and $L(w) \leq D_n^r(w)=l(w) \leq U(w)$ from (19) we have that $|D_{sn}(w)| = l(w)$ and the filter becames

$$H_r = \frac{D_s-l}{D_s-l+D_n^r-l} = 1$$

The integral is

$$I_{b_1} = \int_{b_1} (H_r)^2 (D_n^r - D_n) dw = \int_{b_1} (D_n^r - D_n) . dw$$

<u>case b 2.</u> Again $H_r(w)=1$ and the proof is exactly the same as above.

<u>case b3 .</u> Here $D_n^r(w)=L_n(w) > D_s(w)$ so from (19) $|D_{sn}(w)|^r = D_s(w)$ and the filter is $H_r(w)=0$. Because $D^r_n(w)=L_n(w)$ we have $D^r_n(w)-D_n(w) \leqslant 0$ and the integral is

$$I_{b_3} = \int_{b_3} (H_r)^2 (D_n^r - D_n) dw = 0 \geqslant \int_{b_3} (D_n^r - D_n) dw$$

<u>case b4 .</u> Here $D_n^r(w)=U_n(w) < D_s(w)$ and since

$U(w) > D_s(w) > D^r_n(w)=U_n(w)$ from (19) we have that $|D_{sn}(w)|^r = \max[U_n(w), L(w)]$. The filter is

$$H_r = \frac{D_s - |D_{sn}|^r}{D_s - |D_{sn}|^r + D_n - |D_{sn}|^r} \leqslant 1$$

Because $D_n(w)=U_n(w)$ then $D_n(w)-D_n(w) \geqslant 0$ and the integral is

$$I_{b_4} = \int_{b_4} (H_r)^2 (D_n^r - D_n) dw = \geqslant \int_{b_4} (D_n^r - D_n) dw$$

<u>B3. $L(w) \geqslant D_s(w)$.</u>

Here since $D_s(w) \geqslant L(w)$ it is easy to see that

$| D_{sn}(w) |^r = L(w)$.

<u>case b5.</u> Here we have $D_n(w)=3\cdot L(w)-2D_s(w)$ and by substituting in the expression for the filter

$$H_r = \frac{D_s - L}{D_s - L + D_n^r - L} = -1$$

The integral is

$$I_{b5} = \int_{b5} (H_r)^2 (D_n^r - D_n) dw = \int_{b5} (D_n^r - D_n) dw$$

<u>case b6.</u> For this case we have $D_n(w)=L_n(w) > 3L(w)-2D_s(w)$ and the filter is

$$H_r = \frac{D_s - L}{D_s - L + L_n - L}$$

By replacing $L_n(w)$ with $3L-2D < L_n$ the filter is

$$(H_r)^2 = \left[\frac{D_s - L}{D_s - L + \quad_n - L}\right]^2 \leq 1$$

Because $D^r_n(w)=L_n(w)$ then $D_n^r(w)-D_n(w) \leq 0$ and the integral is

$$I_{b_6} = \int_{b6} (H_r)^2 (D_n^r - D_n) dw \geq \int_{b6} (D_n^r - D_n) dw$$

<u>case b_7.</u> Here $D_n(w)=U_n(w) < 3L(w)-2D(w)$ and the filter is

$$H_r = \frac{D_s - L}{D_s - L + U_n - L}$$

By replacing $D^r_n(w)$ with $3 \cdot L(w) - 2\, D_s(w) > D^r_n(w)$ the filter is

$$[H_r]^2 = \left[\frac{D_s - L}{D_s - L + D^r_n - L}\right]^2 \geq 1$$

And since $D_n(w) = U_n(w)$ we will have $D_n(w) - D_n(w) \geq 0$ so the integral is

$$I_{b7} = \int_{b7} (H_r)^2 (D^r_n - D_n)\, dw \geq \int_{b7} (D^r_n - D_n)\, dw$$

We proved for every case b_i that

$$I_{b_i} \geq \int_{b_i} (D^r_n - D_n)\, dw$$

By adding all the cases we have

$$I = \Sigma I_{b_i} \geq \Sigma \int_{b_i} (D^r_n - D_n)\, dw = 0$$

<u>Subcase C.</u>

<u>C1. $D_s(w) \geq U(w)$.</u>

<u>case c1.</u>

Here $D_n(w) = F(w) > U(w)$ and because $D_s(w) \geq U(w)$ from (19) we have that $|D_{sn}(w)|^r = U(w)$. So the filter is

$$H_r = \frac{D_s - L}{D_s - U + F - L} = \frac{1}{1 + k}$$

The integral becames

$$I_{c_1} = \int_{c_1} (H_r)^2 (D^r_n - D_n)\, dw = \frac{1}{(1+k)^2} \int_{c_1} (D^r_n - D_n)\, dw$$

<u>case c_2.</u> For this case $D_n(w)=L_n(w) > F(w) \geq U(w)$ and again as above $|D_{sn}(w)|^r=U(w)$. The filter is

$$H_r = \frac{D_s-U}{D_s-U+D_n^r-U}$$

If we replace $D_n^r(w)$ with $F(w) > D_n^r(w)$ then the filter is

$$H_r = \frac{D_s-U}{D_s-U+D_n^r-U} \leqslant \frac{1}{(1+k)}$$

Because $D_n(w)=L_n(w)$ we have $D_n^r(w)-D_n(w) \leqslant 0$ and the integral becames

$$I_{c_2} = \int_{c_2} (H_r)^2 (D_n-D_n)\,dw \geq \frac{1}{(1+k)^2} \int_{c_2} (D_n^r-D_n)\,dw$$

<u>case c_3.</u> Here $D_n(w)=U_n(w)$ and because $D_s(w) \geq U(w)$ we have that $|D_{sn}(w)|^r$ is given by

$$|D_{sn}(w)|^r=\text{seclargest}\{U_n(w), U(w), L(w)\}$$

We will prove that

$$k\,D_s(w) + (1-k)\,|D_{sn}(w)|^r \geq U_n(w)$$

If $U_n(w) \geq U(w) \geq L(w)$ then $|D_{sn}(w)|^r=U(w)$ and the above relation becames

$$k\,D_s(w) + (1-k)\cdot U(w) \geq U_n(w)$$

which is true by assumption for case c_2. If $U(w) \geq U_n(w)$ then $|D_{sn}(w)|^r=\max[U_n(w), L(w)]$ and because $D_s(w) \geq U(w) \geq \max[U_n(w), L(w)]$ and $k > 0$ then

$$k\cdot D_s(w)+(1-k)\cdot|D_{sn}(w)|^r \geq |D_{sn}(w)|^r=\max[U_n(w), L(w)] \geq U_n(w)$$

If in the expression for the filter we replace $D_n^r(w)=U_n(w)$

with the above larger expression then we get

$$H_r = \frac{D_s - |D^r_{sn}|}{D_s - |D^r_{sn}| + U_n - |D^r_{sn}|} \geqslant \frac{1}{(1+k)}$$

Because the $D^r_n(w) = U_n(W)$ we will have $D^r_n(w) - D_n(w) \geqslant 0$ and the integral becames

$$I_{c_3} = \int_{c_3} [H_r]^2 [D^r_n - D_n] dw \geqslant \frac{1}{(1+k)^2} \int_{c_3} [D^r_n - D_n] dw$$

<u>C2. $U(w) > D_s(w) \geqslant L(w)$.</u>

<u>case c_4.</u> Here $D_n(w) = D_s(w)$ and because of the assumption for C2 and (19) the $|D_{sn}(w)|^r$ is also equal to $D_s(w)$. For this case the filter is undefined so it can take any value we like and we will give the value

$$H_r = \frac{1}{(1+k)}$$

The integral also is

$$I_{c_4} = \int_{c_4} [H_r]^2 [D^r_n - D_n] dw = \frac{1}{(1+k)^2} \int_{c_4} [D^r_n - D_n] dw$$

<u>case c_5.</u> Because here $D^r_n(w) = L_n(w) > D_s(w)$ again it is easy to see that $|D_{sn}(w)|^r = D_s(w)$ and so the filter becames zero. Because also $D^r_n(w) = L_n(w)$ we have that $D^r_n(w) - D_n(w) \leqslant 0$ and the integral is

$$I_{c_5} = \int_{c_5} [H_r]^2 [D^r_n - D_n] dw = 0 \geqslant \frac{1}{(1+k)^2} \int_{c_5} [D^r_n - D_n] dw$$

case c_6.

Here $D_n(w)=U_n(w) < D_s(w)$ from (19) we have $|D_{sn}(w)|=\max\{L(w),U_n(w)\}$ and $D_n(w)-|D_{sn}(w)|^r \leq 0$. Because $k > 0$ we have for the filter

$$H_r = \frac{D_s - |D_{sn}|^r}{D_s - |D_{sn}|^r + U_n - |D_{sn}|^r} \geq 1 \geq \frac{1}{(1+k)}$$

And since $D_n(w)=U_n(w)$ then $D_n^r(w)-D_n(w) \geq 0$ and the integral is

$$I_c = \int_{c6} [H_r]^2 [D_n - D_n] dw \geq \frac{1}{(1+k)^2} \int_{c6} [D_n - D_n] dw$$

C3. $L(w) > D_s(w)$.

As we have seen many times for this case $|D_{sn}(w)|^r = L(w)$

case c_7.

Here $D_n(w)=G(w)$ and by substituting into the expression for the filter

$$(H_r)^2 = [\frac{D_s - L}{D_s - L + G - L}]^2 = \frac{1}{(1+k)^2}$$

The integral is

$$I_{c7} = \int_{c7} [H_r]^2 [D_n - D_n] dw = \frac{1}{(1+k)^2} \int_{c7} [D_n - D_n] dw$$

case c_8. For this case we can see that

$D_n^r(w)=L_n(w) > G(w)$ so the filter is

$$H_r = \frac{D_s-L}{D_s-L+L_n-L}$$

By replacing $D_n^r(w)$ with $G(w) < D_n^r(w)$ the filter becames

$$[H_r]^2 = [\frac{D_s-L}{D_s-L+L_n-L}]^2 \leqslant [\frac{D_s-L}{D_s-L+G-L}]^2 = \frac{1}{(1+k)^2}$$

Because $D_n^r(w)=L(w)$ we will have $D_n^r(w)-D_n(w) \leqslant 0$ and the integral is

$$I_{c_8} = \int_{c_8}[H_r]^2[D_n^r-D_n]dw \geqslant \frac{1}{(1+k)^2}\int_{c_8}[D_n^r-D_n]dw$$

<u>**case c_9.**</u> Here $D_n^r(w)=U_n(w) < G(w)$ and the filter is

$$H_r = \frac{D_s-L}{D_s-L+U_n-L}$$

By replacing $D_n^r(w)$ with $G(w) > D_n^r(w)$ we get

$$[H_r]^2 = [\frac{D_s-L}{D_s-L+U_n-L}]^2 \geqslant [\frac{D_s-L}{D_s-L+G-L}]^2 = \frac{1}{(1+k)^2}$$

Because $D_n^r(w)=U_n(w)$ then $D_n^r(w)-D_n(w) \geqslant 0$ and the integral is

$$I_{c_9} = \int_{c_9}[H_r]^2[D_n-D_n]dw \geqslant \frac{1}{(1+k)^2}\int_{c_9}[D_n-D_n]dw$$

For all case c_i we have that

$$I_{c_i} \geqslant \frac{1}{(1+k)^2}\int_{c_i}[D_n-D_n]dw$$

and by adding

$$I = \Sigma I_{c_i} \geqslant \frac{1}{(1+k)^2} \int [D_n - D_n] \, dw = 0$$

Subcase D.

case d_1. For this case $D_n(w) = 1(w) \geqslant \max\{D_s(w), L(w)\} \geqslant D_s(w)$ also $U(w) \geqslant D_s(w) \geqslant L(w)$ so from (19) we have that $|D_{sn}(w)|^r = D_s(w)$ and this means that $H_r(w) = 0$.

The integral then is also zero

$$I_{d_1} = \int_{d_1} [H_r]^2 [D_n - D_n] dw = 0$$

case d_2. Here $D^r_n(w) = U_n(w)$ so $D^r_n(w) - D_n(w) \geqslant 0$ and for the integral we have

$$I_{d_2} = \int_{d_2} [H_r]^2 [D_n - D_n] dw \geqslant 0$$

For both cases $I_{d_i} \geqslant 0$ and by adding them

$$I = \Sigma I_{d_i} \geqslant 0$$

And this completes the proof for all possible subcases.

3.4 Other Models.

Since we have finished the presentation of the band model we can talk about some other models and see how we can apply the theory of chapter 2.

P-point model.

In this model we assume knowledge only of the signal and noise power in given sets without assuming any knowledge of the shape of the spectra. Robust filters for this problem for the uncorrelated signal and noise case are presented in [9]. These filters turn out to be piecewise constant. For the correlated signal and noise case we have to assume also knowledge of the integral $\int |D_{sn}(w)|\,dw$ itself or knowledge of bounds of this integral. Following the theory of chapter 2 it is very easy to find the robust filter for this case. It is again piecewise constant.

Assume that a collection of sets Ai is given such that $Ai \cap Aj = \phi$ for $i \neq j$ and that $\bigcup_i Ai = R$ the real line. Also assume that in every set Ai we know that

$$\int_{Ai} D_s(w)\,dw = 2\pi\sigma_{si}^2 \qquad \int_{A_i} |D_{sn}(w)|\,dw \leq 2\pi p_i^2$$

$$\int_{Ai} D_n(w)\,dw = 2\pi\sigma_{ni}^2$$

Obviously the resulting class is convex so we can apply chapter 2. Since we do not know anything about the shape and we expect piecewise constant filter again , we can assume that in each set

$$\frac{D_s(w)}{D_n(w)} = k_i \qquad \text{and} \qquad \frac{|D_{sn}(w)|}{D_n(w)} = v_i$$

where k_i and v_i are constants. We can easily show that

$$k_i = \frac{\sigma_{si}^2}{\sigma_{ni}^2} \qquad \text{and} \quad v_i \leq \frac{p_i^2}{\sigma_{ni}^2} = u_i$$

According to section 2.4 $|D_{sn}(w)|^r$ is the function as close as possible to $\min[D_s(w), D_n(w)]$. Since $D_s(w) = k_i \cdot D_n(w)$ and

$|D_{sn}(w)| \leq u_i \cdot D_n(w)$ we have that $u_i D_n(w)$ behaves as an upper bound for $|D_{sn}(w)|$. So

$|D_{sn}(w)|^r = D_s(w)$ if $D_s(w) \leq \min[D_n(w), u_i \cdot D_n(w)]$ or

$k_i \leq \min[1, u_i]$ or $\sigma^2_{s_i} \leq \min[\sigma^2_{n_i}, p^2_i]$

and here $H_r(w) = 0$. In the same way

$|D_{sn}(w)|^r = D_n(w)$ if $\sigma^2_{n_i} \leq \min[\sigma_{s_i}, p^2_i]$

and $H_r(w)=1$

$|D_{sn}(w)|^r = u_i \cdot D_n(w)$ if $p^2_i \leq \min[\sigma^2_{s_i}, \sigma^2_{n_i}]$

and

$$0 \leq H_r = \frac{\sigma^2_{s_i} - p^2_i}{\sigma^2_{s_i} - p^2_i + \sigma^2_{n_i} - p^2_i} \leq 1$$

To prove now that this filter is the robust filter for this model, we have to show that

$$e_{op}(D^r) = e(D^r, H_r) \geq e(D, H_r)$$

or because of (14)

$$\int [1-H_r]^2 [D^r_s - D_s] dw + \int [H_r]^2 [D^r_n - D_n] dw + 2\int [H_r][1-H_r][Re(D_{sn}) - Re(D^r_{sn})] dw \geq 0$$

The first two integrals are zero because in each set the filter is constant and $D^r_i(w), D_i(w), i=s,n$ have the same power. For the third integral whenever $|D_{sn}(w)|^r = D_s(w)$ $H_r(w)=0$ so the integral is also zero. When $|D_{sn}(w)|^r = D_n(w)$ then $H_r(w)=1$ and the integral is again zero. When $|D_{sn}(w)|^r = u_i D_n(w)$ then $H_r(w)$ is a constant between zero and one. Also $D^r_{sn}(w) = -|D_{sn}(w)|^r$ $\qquad Re[D^r_{sn}(w)] = -|D_{sn}(w)|^r$

The integral then is

$$\int [H_r]\ [1-H_r][Re(D_{sn})-Re(D^r_{sn})]dw = \sum \int_{A_i} [H_{r_i}][1-H_{r_i}][Re(D_{sn})-Re(D^r_{sn})]dw$$

$$\sum [H_{r_i}][1-H_{r_i}]\ [p_i^2 + \int_{A_i} Re(D_{sn}) \cdot dw\ ] \geq 0$$

Thus

$$e(D^r, H_r) \geq e(D, H_r)$$

CHAPTER 4

Numerical Examples.

Example 1.

Assume that signal and noise are given as in figure 13. Assume also that there is no restriction on the $D_{sn}(w)$.

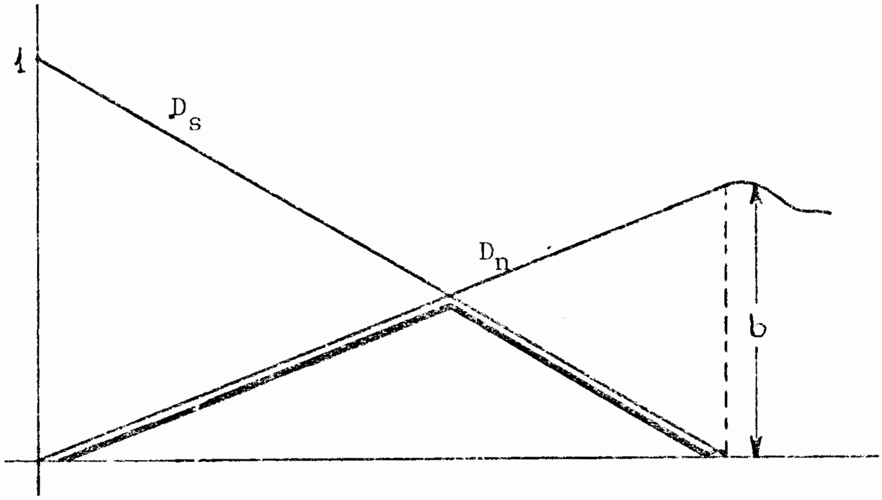

Figure 13.

For this case according to section 3.1(A) the robust filter is given:

$$H_r(w) = \begin{cases} 1 & \text{when } D_s(w) \geqslant D_n(w) \\ 0 & \text{otherwise} \end{cases}$$

Also we proved that this filter has the same performance for any $D_{sn}(w)$ in the MSE sense and we will call this error e_r. Now if we design the Wiener filter assuming wrongly that signal and noise are uncorrelated this filter will be:

$$H_u(w) = \frac{D_s(w)}{D_s(w)+D_n(w)}$$

If we apply H_u to any matrix D the resulting MSE takes its maximum value for $D_{sn}(w)=-\sqrt{D_s(w)\ D_n(w)}$ and we will call this error e_u. If signal and noise are indeed uncorrelated then H_u is the optimum filter and the resulting error we will call it e. In the following table for different values of the parameter b we calculate e_r the performance for H_r, e_u the worst performance for H_u, e the best performance for H_u and the percentage of improvement of performance of e_r over e_u.

b	e_r	e_u	%	e
0.1	0.09	0.12	34.5	0.07
0.5	0.33	0.42	27.1	0.23
1.0	0.5	0.62	25.7	0.33
3.0	0.75	0.97	29.0	0.53
5.0	0.83	1.10	31.7	0.62
10.0	0.91	1.20	34.5	0.73
15.0	0.95	1.28	34.8	0.81

From the table we can see that H_r can indeed be considerably better for this particular example. Even if signal and noise are uncorrelated e_r is not far from e.

But H_r is also a significantly simpler filter since it is only 0 or 1.

Example 2.

Assume that the following bounds for signal and noise are given:

$$U_s(w) = \begin{cases} 2.5 & \text{for } 0 \leqslant w \leqslant 5. \\ 0 & \text{otherwise} \end{cases} \qquad L_s(w) = \begin{cases} 2.0 & 0 \leqslant w \leqslant 5 \\ 0 & \text{otherwise} \end{cases}$$

$$U_n(w) = \frac{16}{w^2 + 4} \qquad L_n(w) = \frac{12}{w^2 + 4}$$

also:

$$\int_{\infty}^{\infty} D_s(w) . dw = 24 \qquad \int_{\infty}^{\infty} D_n(w) . dw = 21$$

The bounds are illustrated in the following figure.

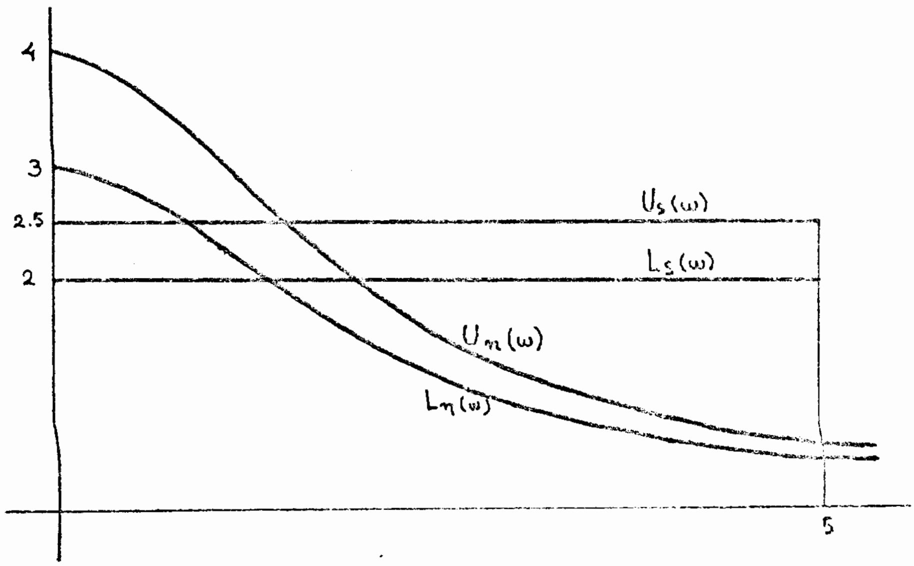

Figure 14.

We assume no restriction on $|D_{sn}(w)|$.

We can easily calculate :

$\int_{\infty}^{\infty} U_s(w).dw=25$ $\int_{\infty}^{\infty} U_n(w).dw=25.133$

$\int_{\infty}^{\infty} B_s(w).dw=21.765$ $\int_{\infty}^{\infty} B_n(w).dw=21.382$

$\int_{\infty}^{\infty} A_s(w).dw=21.155$ $\int_{\infty}^{\infty} A_n(w).dw=20.772$

$\int_{\infty}^{\infty} L_s(w).dw=20$ $\int_{\infty}^{\infty} L_n(w).dw=18.85$

So we have here the case A7 of section 3.2 where $2\pi\sigma_s^2 > \int_{\infty}^{\infty} B_s(w)dw$ and $\int_{\infty}^{\infty} B_n(w)dw \geqslant 2\pi\sigma_n^2 > \int_{\infty}^{\infty} A_n(w)dw$. The l.f.pair of signal and noise are shown below in figure 15.

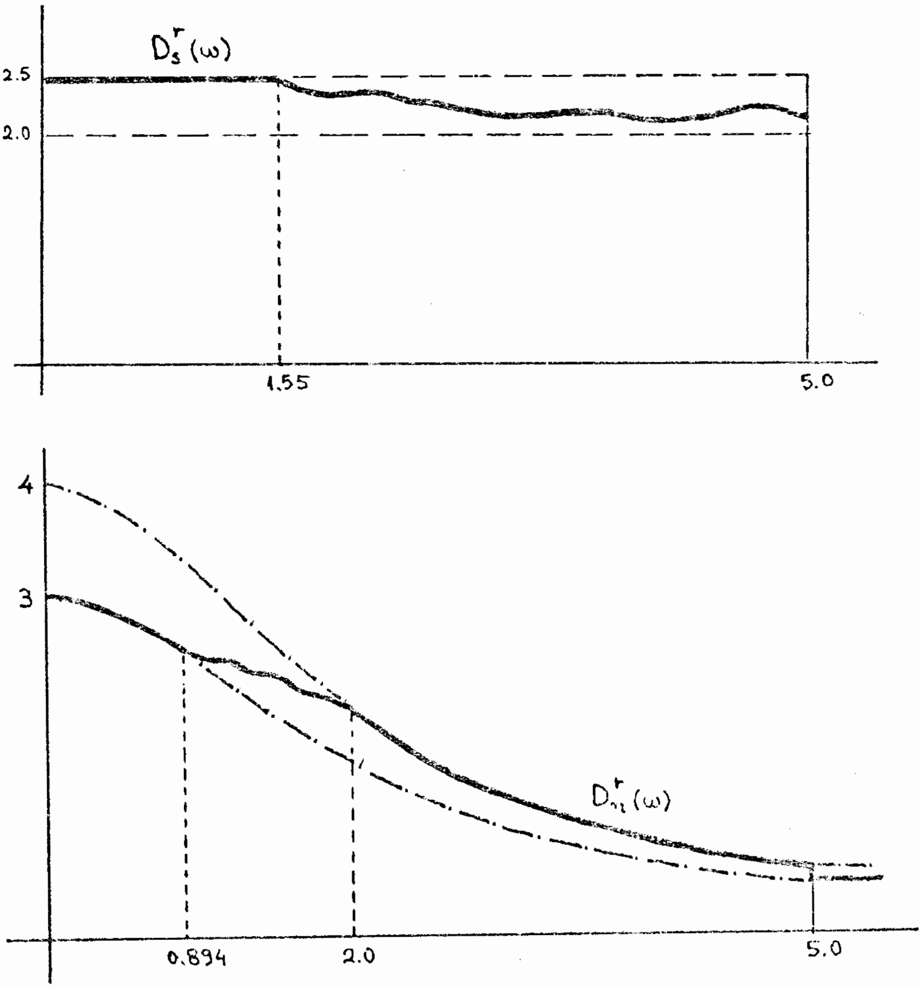

Figure 15.

Using the formula for the error given in case A the MSE error is:

$$e_{op}(D^r)=2.168$$

Let us now compare this result to the regular Wiener filtering. We are going to select two nominal densities one for signal and one for noise that satisfy the power constraints and the boundaries. Such densities are

$$D_s(w)=\begin{cases} 2.4 & \text{for } 0\leq w\leq 5 \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad D_n(w)=\frac{\frac{44}{\pi}}{w^2+4}$$

and they are illustrated in the following figure.

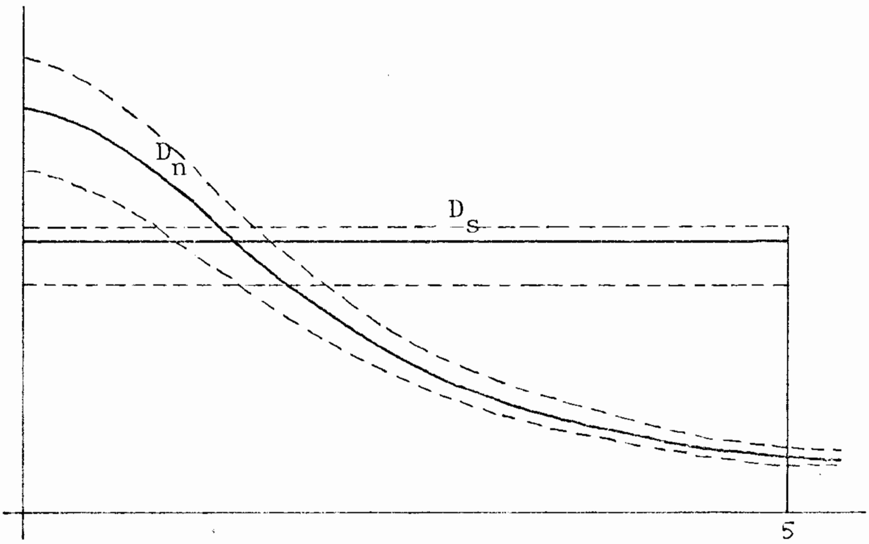

Figure 16.

If we assume that signal and noise are uncorrelated and design the Wiener filter $H_u(w)$ then the MSE when we apply $H_u(w)$ takes its maximum value for $-D_{sn}(w)=\sqrt{D_s(w)\cdot D_n(w)}$.

This value is e_u =2.829. Compaired to the robust filter MSE upper bound the robust filter has at least 30.5 % better performance.

CHAPTER 5

Conclusion.

With this work we applied the idea of robust filters to the correlated signal and noise case. We stated and proved a theorem and we applied it to find the solution to some very interesting classes of matrices. In practice it is very common to find cases where signal and noise are correlated, and there we can apply these filters.

As we have seen the filters we present here are non-causal and they cannot be used for real time processes. But there are many applications where causality has no meaning. Image processing is one such application. Also in array design , arrays are treated as spatial filters where causality again has no meaning.

Topics for further investigation.

In chapter 3 we present the various classes. All of them are special cases of the general band model. The solution for this general problem is not given , but using the results from section 3.3 as a guide we can find the solution.

For the uncorrelated signal and noise there exist several distance measures that indicate how different the signal and the noise are. They are of the form

$$\int C(x) \, . \, D_s(w) dw$$

where C(x) a convex function of x and $x=\frac{D_s(w)}{D_n(w)}$. If there exist classes for $D_s(w)$ and $D_n(w)$, then under general conditions there exist a pair $D_s^r(w)$, $D_n^r(w)$ that has the smallest distance. This pair is independent of the actual form of C(x). It is possible to extend this idea to the correlated signal and noise case. We can define a measure

$$\int C(p) \cdot D_s(w)\,dw$$

Where C(p) is a convex function of p and $p=\frac{|D_{sx}(w)|^2}{D_s(w) \cdot D_x(w)}$

This measure will indicate the amount of information about s(t) that exist in x(t)=s(t)+n(t). Given a class of density matrices it might be possible to find a matrix that has the maximum amount of information regardless of the actual form of the C(p). Since the functional $e_{op}(D)$ is of the above form, the least favorable matrix is the most possible candidate for the matrix with the maximum amount of information.

APPENDIX

Before proving theorem 0 we will state and prove the following lemma.

<u>Lemma.</u>

Let Δ be convex and for D' and $D'' \in \Delta$, define $D = (1-\varepsilon)D' + \varepsilon D''$ with $0 \leqslant \varepsilon \leqslant 1$. Then the expression:

$$G(\varepsilon, w) = \frac{D_s^\varepsilon(w) \cdot D_x^\varepsilon(w) - |D_{sx}^\varepsilon(w)|^2}{D_x^\varepsilon(w)}$$

is a convex function of ε.

<u>Proof.</u>

First we will show that:

$$\frac{D_s^\varepsilon \cdot D_x^\varepsilon - |D_{sx}^\varepsilon|^2}{D_x} \geqslant (1-\varepsilon)\frac{D_s' \cdot D_x' - |D_{sn}'|^2}{D_x'} + \varepsilon \frac{D_s'' \cdot D_x'' - |D_{sx}''|^2}{D_x''} \qquad (20)$$

Subtracting each side from $D_s(w)$ it is equivalent to show that:

$$\frac{|D_{sx}^\varepsilon|^2}{D_x^\varepsilon} \leqslant (1-\varepsilon)\frac{|D_{sx}'|^2}{D_x'} + \varepsilon \frac{|D_{sx}''|^2}{D_x''} \quad \text{or}$$

$$|(1-\varepsilon)D_{sx}' + \varepsilon D_{sx}''|^2 \leqslant \left\{(1-\varepsilon)\frac{D_{sx}'}{D_x} + \varepsilon \frac{D_{sx}''}{D_x''}\right\} \cdot \left\{(1-\varepsilon)D_x' + \varepsilon D_x''\right\}$$

But this is the Schwartz inequality. To prove now that $G(\varepsilon, w)$ is a convex function of ε we have to prove that for any ε_1, ε_2 and $0 \leqslant a \leqslant 1$

$$\frac{D_s^u \cdot D_x^u - D_{sx}^u}{D_x^u} \geqslant (1-a)\frac{D_s^{\varepsilon 1} \cdot D_x^{\varepsilon 1} \; D_{sx}^{\varepsilon 1}}{D_x^{\varepsilon 1}} + a\frac{D_s^{\varepsilon 2} \cdot D_x^{\varepsilon 2} \; D_{sx}^{\varepsilon 2}}{D_x^{\varepsilon 2}}$$

where $u=(1-a)\varepsilon_1+a\varepsilon_2$.

Since (20) is true for any D',D'' and any ε it is also true for $D'=D^{\varepsilon_1}, D''=D^{\varepsilon_2}$ and $\varepsilon=a$ so by applying (20) we get exactly what we want.

Proof of Theorem 0.

The only if part is easy. From (10) we have that:

$$e_{op}(D,H_r) \leq e(D^r,H_r) \leq e_{op}(D^r)$$

but

$$e_{op}(D) \leq e(D,H^r) \quad \text{so}$$

$$e_{op}(D) \leq e_{op}(D^r)$$

To prove the if part, define $D^\varepsilon=(1-\varepsilon)D^r+\varepsilon D$ where $0\leq\varepsilon\leq 1$ and $D \in \Delta$. Because of the previous lemma and the fact that

$$e_{op}(D^\varepsilon) = \int_{\infty}^{\infty} G(\varepsilon,w).dw$$

$e_{op}(D^\varepsilon)$ is also a convex function of ε. So:

$$e_{op}(D^\varepsilon) \geq (1-\varepsilon)e_{op}(D^r)+\varepsilon e_{op}(D)$$

and

$$0 \geq \frac{e_{op}(D^\varepsilon)-e_{op}(D^r)}{\varepsilon} \geq e_{op}(D)-e_{op}(D^r) \qquad (21)$$

From the convexity of $e_{op}(D^\varepsilon)$ we conclude that

$$\frac{e_{op}(D^\varepsilon)-e_{op}(D^r)}{\varepsilon}$$

is monotonic with respect to ε. Because of (21) is also bounded so the limit

$$\lim_{\varepsilon\to 0^+} \frac{e_{op}(D\ )-e_{op}(D^r)}{\varepsilon}$$

exists. From the convexity of $G(\varepsilon,w)$, $\frac{dG(\varepsilon,w)}{d\varepsilon}$ is also a monotonic function with respect to ε. These conditions allow us to write the following:

From (21)

$$0 \geq \lim_{\varepsilon\to 0^+} \frac{1}{2\pi}\int \frac{G(\varepsilon,w)-G(0,w)}{\varepsilon}dw = \frac{1}{2\pi}\int \lim_{\varepsilon\to 0^+}\frac{G(\varepsilon,w)-G(0,w)}{\varepsilon}dw$$

$$\geq \frac{1}{2\pi}\int \left.\frac{d\ G(\varepsilon,w)}{d\varepsilon}\right|_{\varepsilon=0^+} dw \qquad (22)$$

But

$$\left.\frac{dG(\varepsilon,w)}{d\varepsilon}\right|_{\varepsilon=0^+} = \{D_s - \frac{D^*_{sx}D^r_{sx}+(D^r_{sx})^*D_{sx}}{D^r_x} + \frac{D^r_{sx}}{D^r_x}D_x\} - \{D^r_s - \frac{D^r_{sx}}{D_x}\} \qquad (23)$$

Because D_x is even function of w and $Im(D^r_{sx}, D^*_{sx})$ is odd we have

$$\int \frac{D^r_{sx}.D^*_{sx}}{D^r_x}dw = \int\frac{(D^r_{sx})^*.D_{sx}}{D^r_x}dw$$

So by integrating equation (23) we have that

$$\int \left.\frac{dG(\varepsilon,w)}{d\varepsilon}\right|_{\varepsilon=0^+} dw = e(D,H_r) - e_{op}(D^r)$$

And because of (22) $\quad 0 \geq e(D,H_r) - e_{op}(D^r)$

And this concludes the proof.

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