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F-35042 Rennes Cédex, France

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Georges V. MOUSTAKIDES

IRISA

INSTITUT DE RECHERCHE EN INFORMATIQUE
ET SYSTÈMES ALÉATOIRES

Campus Universitaire de Beaulieu
Avenue du Général Leclerc
35042 - RENNES CÉDEX
FRANCE
Tél. : (99) 36.20.00
Télex : UNIRISA 95 0473 F

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ROBUST DETECTION OF SIGNALS
A LARGE DEVIATIONS APPROACH

George V Moustakides

SUMMARY

Robust detection of a signal is considered for the case of i.i.d. observations. Following an asymptotic but non-local approach, the exponential rates of decrease of the error probabilities are considered as measure of performance. Under this measure a robust detection structure for the symmetric density case is derived. This detection structure is a generalization of an existing result for the local case and is reduced to it when the signal magnitude tends to zero.

RESUME

On considère le problème de la detection robuste dans le cas d' observations independantes. A l' aide d' une méthode asymptotique nonlocale, on établit la vitesse de décroissance exponentielle de la probabilité d' erreur. A partir de cette mesure, on donne un detecteur robuste pour le cas de densités symétriques. Cette nouvelle méthode de detection est la généralisation d' une méthode existante dans le cas local, et tend vers cette dernière lorsque le signal devient de plus en plus faible.

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$$-\lim_{n\to\infty}\frac{1}{n}\log P_n^1(\psi_o) \geq -\lim_{n\to\infty}\frac{1}{n}\log P_n^1(\psi) \tag{5}$$

The parameter α is known as the exponential level of the test and the rate of decrease of the probability $P_n^1(\psi)$ as the exponential power. It is easy to see that the exponential power can play here the same role as the efficacy in the local case. Indeed if n_1, n_2 are the number of observations required by two different tests to reach the same power, then we have

$$\frac{-\lim\limits_{n_1\to\infty}\frac{1}{n_1}\log P_{n_1}^1(\psi_1)}{-\lim\limits_{n_2\to\infty}\frac{1}{n_2}\log P_{n_2}^1(\psi_2)} = \lim_{n_1,n_2\to\infty}\frac{n_1}{n_2} \tag{6}$$

We now give a theorem that defines the optimum nonlinearity in the sense of (4) and (5).

Theorem 1. Let $f_0(x)$ and $f_1(x)$ be two densities with the same support, then the optimum nonlinearity $\psi_o(x) \in \Psi_\alpha$ in the sense of (4) and (5) is given by the log-likelihood

$$\psi_o(x) = \log\frac{f_1(x)}{f_0(x)} \tag{7}$$

and γ is defined in such a way that (4) is satisfied with equality.

Proof. The proof is given in [9, page 158]. Actually we can prove a much stronger result, we can prove that the test defined with Theorem 1 has the largest exponential power among all tests of exponential level α and not only of those of the form of (2).

We now present a theorem that defines more explicitly the two rates, for a test of the form of (2), in terms of the two densities and the nonlinearity $\psi(x)$.

Theorem 2. Let $f_0(x)$ and $f_1(x)$ be two densities with the same support, let $\psi(x)$ be a nonlinearity which is integrable with respect to

$f_0(x)$ and $f_1(x)$. Let also γ be a real number that satisfies

$$E_1\{\psi(x)\} > \gamma > E_0\{\psi(x)\} \tag{8}$$

then we have for the two rates that

$$A_0(\psi,f_0) = -\lim_{n\to\infty}\frac{1}{n}\log P_n^0(\psi) = -\log\min_{r\geq 0} E_0\{e^{r[\psi(x)-\gamma]}\}$$

$$A_1(\psi,f_1) = -\lim_{n\to\infty}\frac{1}{n}\log P_n^1(\psi) = -\log\min_{r\geq 0} E_1\{e^{r[\gamma-\psi(x)]}\} \tag{9}$$

Proof. The proof is in the Appendix. The threshold γ must satisfy (8) in order to have exponential decrease for the two error probabilities and validity of (9). This requirement bounds the possible values of the exponential level. We can see from (9) that the exponential level is increasing with γ thus, the maximum value it can take is when $\gamma = E_1\{\psi(x)\}$. For this value the error probability under H_1 has rate equal to zero, i.e. we do not have exponential decrease. We now apply these results to the robust detection theory.

III. ROBUST DETECTION.

Let $N_1, N_2, \ldots$ be a noise sequence with common density $f(x)$. We would like to decide between the following two hypotheses

$$H_0: \quad X_i = N_i + s_0 \qquad s_0 \in (-\infty, 0] \quad i=1,2,\ldots$$

$$H_1: \quad X_i = N_i + s_1 \qquad s_1 \in [s,\infty) \quad i=1,2,\ldots$$

where $\{X_i\}$ is the observation sequence, s_0, s_1 are assumed unknown, but $s > 0$ is assumed known.

Let F be the class of all symmetric densities that satisfy the following ε-contamination model

$$f(x) = (1-\varepsilon)g(x) + \varepsilon h(x) \tag{11}$$

where $0 < \varepsilon < 1$ is known, $g(x)$ is a known symmetric strongly unimodal, nowhere vanishing density and $h(x)$ is an unknown symmetric density. Let Ψ_α denote the class of all nonlinearities $\psi(x)$ for which there exist a test of the form of (2) satisfying for every $f(x) \in F$ the following

$$A_0(\psi, f) \geq \alpha \tag{12}$$

We would like to find a density $f_l(x) \in F$ and a $\psi_r(x) \in \Psi_\alpha$ such that

$$A_1(\psi_r, f) \geq A_1(\psi_r, f_l) \geq A_1(\psi, f_l) \tag{13}$$

and also

$$A_0(\psi_r, f) \geq A_0(\psi_r, f_l) = \alpha \tag{14}$$

The right inequality of (13) and the right equality of (14), using Theorem 1, suggest that $\psi_r(x)$ is the following log-likelihood ratio

$$\psi_r(x) = \log \frac{f_l(x+s_1)}{f_l(x+s_0)} \tag{15}$$

for some s_1 and s_0. We now define the density $f_l(x)$.

$$f_l(x) = \begin{cases} (1-\varepsilon)g(x) & \text{for} \quad 0 < x < x_0 \\ \dfrac{(1-\varepsilon)}{k^n} g(x-ns) & \text{for} \quad x_0+(n-1)s \leq x \leq x_0+ns \quad n=1,2,\ldots \end{cases} \tag{16}$$

where $k = g(x_0-s)/g(x_0)$ and $x_0 > s/2$ is selected in order to have

$$\int_0^\infty f_l(x)dx = 0.5 \tag{17}$$

A typical form of $f_l(x)$ is given in Figure 1. In the appendix it is shown that an

x_0 always exists and that it is unique and also that $f_l(x) \in F$. From the definition in (16) notice that $\varepsilon h_l(x) = f_l(x) - (1-\varepsilon)g(x)$ puts all its mass outside the interval $(-x_0, x_0)$. Let us now see what is the form of $\psi_r(x)$ defined by (15) when $f_l(x)$ is defined by (16) and $s_0 = 0$, $s_1 = s$.

$$\psi_r(x) = \begin{cases} \log \dfrac{g(x_0-s)}{g(x_0)} & \text{for } x \geq x_0 \\ \log \dfrac{g(x-s)}{g(x)} & \text{for } x_0 \geq x \geq -x_0+s \\ -\log \dfrac{g(x_0-s)}{g(x_0)} & \text{for } -x_0+s \geq x \end{cases} \tag{18}$$

With the following theorem we prove that $f_l(x)$ and $\psi_r(x)$ is the pair that satisfies (13) and (14).

Theorem 3. Let $f_l(x)$ and $\psi_r(x)$ be defined by (16) and (18) then they satisfy (13) and (14).

Proof. The proof is given in the Appendix. As we can see in Figure 1 the least favorable density $f_l(x)$ repeats a piece of length s of the density $g(x)$ after dividing it every time with the constant k. By taking $s \to 0$ the density $f_l(x)$ tends to the density defined by Huber for the local case [1,8].

IV. EXAMPLES.

As an example we present the Gaussian nominal case. Clearly the robust nonlinearity $\psi_r(x)$ will be linear inside the interval $[-x_0+s, x_0]$. In Table I values of x_0 are given for different values of the contamination ε and the signal s. Table II contains the exponential level and the worst case exponential power for the case $s = 1$ and for different values of γ and ε. It is assumed that the two means of $\psi_r(x)$ have been normalized to zero under H_0 and to unity under H_1 and that γ takes values in the interval $[0, 0.5]$. For values of

γ in the interval $[0.5, 1]$ the table is symmetric in the following way: the exponential level at $\gamma > 0.5$ is equal to the worst exponential power at $1-\gamma$ and the worst exponential power is equal to the exponential level.

V. CONCLUSION.

We have presented here a detection structure which is robust to partial knowledge of the signal magnitude and of the noise distribution function. The result is asymptotic but non-local. The advantage of this approach is that the robust detector is a likelihood ratio for a specific density and is always non-trivial, something which is not true for all existing approaches. It will be interesting to see if this approach also applies to the case where the densities are symmetric only inside an interval around the origin, thus generalizing the result in [3].

V. APPENDIX.

Proof of Theorem 2. The proof is based on the fact that the Chernoff bound is asymptotically correct. Thus if $h_0(x)$ and $h_1(x)$ are two densities that have the same support, then we have

$$\lim_{n\to\infty}\frac{1}{n}\log P_0\{\sum_{j=1}^{n}\log\frac{h_1(x)}{h_0(x)} > 0\} = \log\min_{s\in[0,1]}\int h_0^s(x)h_1^{1-s}(x)dx \qquad (19)$$

For a proof see [9, pages 158-161]. In order now to apply this result to our case, let $c > 0$ be the constant that satisfies

$$\int e^{c[\psi(x)-\gamma]}f_0(x)dx = 1 \qquad (20)$$

The constant c exists when γ is between the essential supremum of $\psi(x)$

and its mean. In our case by assumption we have that $f_0(x)$ and $f_1(x)$ have the same support, thus using also (3) we have

$$essup_0\psi(x) = essup_1\psi(x) \ge E_1\{\psi(x)\} > \gamma > E_0\{\psi(x)\} \qquad (21)$$

and thus there always exist a c satisfying (20). Now let us define

$$f_1'(x) = e^{c[\psi(x)-\gamma]}f_0(x) \qquad (22)$$

If we apply (19) for $f_1'(x)$ and $f_0(x)$ it yields exactly (9) for the probability of error $P_n^0(\psi)$ with $r = sc$. In a similar way we can prove the second equality.

Proof that $f_l(x)\in F$. We first prove existence of an $x_0 > \frac{s}{2}$ that satisfies (17). Define as $B(x_0)$ the following function

$$B(x_0) = \frac{1}{1-\varepsilon}\int_0^\infty f_l(x)dx = \int_0^{x_0}g(x)dx + \frac{1}{l(x_0)-1}\int_{x_0-s}^{x_0}g(x)dx \qquad (23)$$

where $l(x) = g(x-s)/g(x)$. Because of the strong unimodality of $g(x)$ the function $l(x)$ is strictly increasing and thus $l(x) > 1$ for $x > s/2$. Notice that

$$\lim_{x_0\to s/2} B(x_0) = \infty > \frac{1}{2(1-\varepsilon)}$$
$$\lim_{x_0\to\infty} B(x_0) = \frac{1}{2} < \frac{1}{2(1-\varepsilon)} \qquad (24)$$

Using continuity arguments there exists an x_0 that satisfies $B(x_0) = 1/2(1-\varepsilon)$. The uniqueness can be easily deduced by taking the derivative of $B(x_0)$, this derivative is always negative for $x_0 > s/2$.

To prove now that $f_l(x)$ belongs to the class F it is enough to show that

$$f_l(x) \geq (1-\varepsilon)g(x) \tag{25}$$

This inequality is trivial for the case $0 \leq x \leq x_0$. For the case now $x_0 + (n-1)s \leq x \leq x_0 + ns$ it is equivalent, using (16), to

$$\frac{-\log g(x_0-s) + \log g(x_0)}{s} \leq \frac{-\log g(x-ns) + \log g(x)}{ns} \tag{26}$$

and since $-\log g(x)$ is convex and $x - ns \geq x_0 - s$ the inequality in (26) is true.

Proof of Theorem 3. Before proving that $f_l(x), \psi_r(x)$ satisfy (13) and (14) we first prove a lemma.

Lemma 1. Let $\omega(x)$ be a non-decreasing function such that $\omega(x) + \omega(-x)$ is also non-decreasing for $x \geq 0$. If $f(x) \in F$, $s_0 \leq 0$ and $s_1 \geq s$, then

$$\begin{aligned} &i. \int_{-\infty}^{\infty} \omega(\psi_r(x)) f_l(x)dx \geq \int_{-\infty}^{\infty} \omega(\psi_r(x)) f(x-s_0)dx \\ &ii. \int_{-\infty}^{\infty} \omega(\psi_r(x)) f_l(x-s)dx \leq \int_{-\infty}^{\infty} \omega(\psi_r(x)) f(x-s_1)dx \end{aligned} \tag{28}$$

Proof. We only prove the first inequality since in a similar way we can prove the second. Notice first some important properties of the function $\psi_r(x)$ defined in (18). It is non-decreasing with x, the function $\psi_r(x+\frac{s}{2})$ is odd symmetric non-decreasing and for $x \geq 0$ it is non-negative. Notice also that the density $h_l(x)$ puts all its mass on points where $\psi_r(x)$ is maximum. Since $\psi_r(x)$ and $\omega(x)$ are non-decreasing their composition is also non-decreasing, thus

$$\int_{-\infty}^{\infty} \omega(\psi_r(x)) f(x-s_0)dx = \int_{-\infty}^{\infty} \omega(\psi_r(x+s_0)) f(x)dx$$

$$\leq \int_{-\infty}^{\infty} \omega(\psi_r(x)) f(x)dx \tag{29}$$

Using (29) in order to prove $i.$ it is enough to prove the following

$$\int_{-\infty}^{\infty} \omega(\psi_r(x)) f_l(x)dx \geq \int_{-\infty}^{\infty} \omega(\psi_r(x)) f(x)dx \tag{30}$$

or by eliminating common terms

$$\int_{-\infty}^{\infty} \omega(\psi_r(x)) h_l(x)dx \geq \int_{-\infty}^{\infty} \omega(\psi_r(x)) h(x)dx \tag{31}$$

Since $h_l(x)$ puts its mass on points where $|\psi_r(x)|$ is maximum we can see that (31) is equivalent to

$$\frac{\omega(M) + \omega(-M)}{2} \geq \int_{-\infty}^{\infty} \omega(\psi_r(x)) h(x)dx \tag{32}$$

where M is the maximum value of $\psi_r(x)$. Notice now that

$$\int_{-\infty}^{\infty} \omega(\psi_r(x)) h(x)dx \leq \int_{-\infty}^{\infty} \omega(\psi_r(x+\frac{s}{2})) h(x)dx$$

$$= \int_{0}^{\infty} \left[\omega(\psi_r(x+\frac{s}{2})) + \omega(-\psi_r(x+\frac{s}{2}))\right] h(x)dx \leq \frac{\omega(M) + \omega(-M)}{2} \tag{33}$$

The last equality comes from the fact that $\psi_r(x+\frac{s}{2})$ is odd symmetric. The last inequality is true because $\omega(x) + \omega(-x)$ is by assumption non-decreasing for $x > 0$ and $\psi_r(x+\frac{s}{2})$ is nonnegative for $x > 0$. Thus (32) is true.

To prove Theorem 3 we apply Lemma 1. Selecting $\omega(x) = x$ we have from *i.* and *ii.* in (28) that $\psi_r(x)$ has the maximum mean for $f_l(x)$ under H_0 and the minimum under H_1. This is important because if we take the threshold γ between these two means then we are assured that we will have exponential decrease for both errors for any density $f(x) \in F$. To show now the inequalities in (13) and (14) we first show that they are equivalent. Notice that in order to show any of the two, using (9), it is enough to show that for any $r \geq 0$ we have

$$\int_{-\infty}^{\infty} e^{-r\psi_r(x)} f(x-s_1)dx \leq \int_{-\infty}^{\infty} e^{-r\psi_r(x)} f_l(x-s)dx$$
$$\int_{-\infty}^{\infty} e^{r\psi_r(x)} f(x-s_0)dx \leq \int_{-\infty}^{\infty} e^{r\psi_r(x)} f_l(x)dx \tag{34}$$

By change of variables and using the symmetries of $\psi_r(x)$ and $f(x)$ we can see that

$$\int_{-\infty}^{\infty} e^{-r\psi_r(x)} f(x-s_1)dx = \int_{-\infty}^{\infty} e^{r\psi_r(x)} f(x-s_0')dx \tag{35}$$

where $s_0' = s - s_1 \leq 0$. Thus the first inequality is equivalent to the second. The second inequality is true by a simple application of Lemma 1 with $\omega(x) = e^{rx}$. And this concludes the proof.

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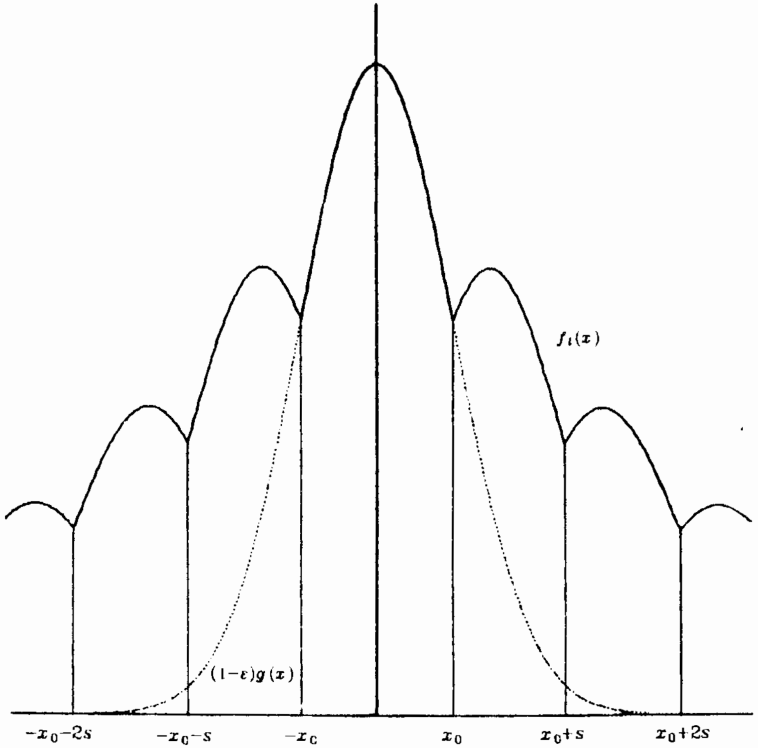

FIGURE 1.

s \ ε	0.001	0.01	0.1	0.5
0.1	2.682	1.994	1.190	0.486
0.5	2.856	2.175	1.378	0.682
1.0	3.032	2.369	1.595	0.919
2.0	3.297	2.684	1.980	1.374
5.0	3.973	3.536	3.080	2.717
10.0	5.760	5.529	5.294	5.110

TABLE 1.

γ \ ε	0.001	0.01	0.1	0.5
0.0	0.000 0.492	0.000 0.459	0.000 0.318	0.000 0.078
0.1	0.005 0.391	0.005 0.372	0.003 0.258	0.001 0.063
0.2	0.020 0.315	0.019 0.295	0.013 0.204	0.003 0.050
0.3	0.045 0.241	0.042 0.226	0.030 0.157	0.007 0.038
0.4	0.079 0.178	0.075 0.167	0.052 0.116	0.013 0.028
0.5	0.124 0.124	0.116 0.116	0.081 0.081	0.020 0.020

TABLE II.