Some New Developments in Sequential Analysis

- Extension of Optimality of Well Known Stopping Times
- Extension of Wald's First Identity to Markov Processes

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Extension of Optimality of Well Known Stopping Times

Given sequentially $\xi_1, \xi_2, \ldots, \xi_n, \ldots$

$\{\mathcal{F}_n\}$ the corresponding filtration

Given conditional probability measures

$\{P_n(\xi_n|\mathcal{F}_{n-1})\}$, $\{Q_n(\xi_n|\mathcal{F}_{n-1})\}$

with $Q_n(\xi_n|\mathcal{F}_{n-1}) \ll P_n(\xi_n|\mathcal{F}_{n-1})$

Hypotheses Testing

$H_0:\quad \{\xi_n\}$ statistics according $\{P_n(\xi_n|\mathcal{F}_{n-1})\}$

$H_1:\quad \{\xi_n\}$ statistics according $\{Q_n(\xi_n|\mathcal{F}_{n-1})\}$

Decide between H_0 and H_1

$\boxed{\text{Stopping Time } N \text{ and decision rule } d_N}$

Disruption

$\{\xi_n\}_1^{m-1}$ statistics according $\{P_n(\xi_n|\mathcal{F}_{n-1})\}$

$\{\xi_n\}_m^{\infty}$ statistics according $\{Q_n(\xi_n|\mathcal{F}_{n-1})\}$

Detect unknown disruption time m

$\boxed{\text{Stopping time } N}$

Optimum Schemes

For $\{\xi_n\}$ i.i.d.

$\{P_n(\xi_n|\mathcal{F}_{n-1})\} = P(\xi_n)$

$\{Q_n(\xi_n|\mathcal{F}_{n-1})\} = Q(\xi_n)$

$$\boxed{l_n = \frac{dQ(\xi_n)}{dP(\xi_n)}}$$

Hypotheses Testing: SPRT

Disruption: Geometric prior

CUSUM

Shiryayev-Roberts

$\boxed{\textbf{All proofs need } \{l_n\} \textbf{ to be i.i.d. and not } \{\xi_n\}}$

Given $\{P_n(\xi_n|\mathcal{F}_{n-1})\}$, $\{Q_n(\xi_n|\mathcal{F}_{n-1})\}$

$$l_n = \frac{dQ_n(\xi_n|\mathcal{F}_{n-1})}{dP_n(\xi_n|\mathcal{F}_{n-1})}$$

If, for all n,

$$P_n\{l_n \leq x|\mathcal{F}_{n-1}\} = F_0(x)$$

then

$$Q_n\{l_n \leq x|\mathcal{F}_{n-1}\} = F_1(x) = \int_0^x z dF_0(z)$$

and $\{l_n\}_i^j$ is i.i.d. under both measures induced by the two sequences of conditional measures.

Examples

Finite State Markov Chains

Two States:

$$P = \begin{bmatrix} p & 1-p \\ 1-p & p \end{bmatrix} \quad Q = \begin{bmatrix} q & 1-q \\ 1-q & q \end{bmatrix}$$

$$L = \begin{bmatrix} \frac{q}{p} & \frac{1-q}{1-p} \\ \frac{1-q}{1-p} & \frac{q}{p} \end{bmatrix} \quad P = \begin{bmatrix} p & 1-p \\ 1-p & p \end{bmatrix}$$

$$P(l_n = \tfrac{q}{p} | \mathcal{F}_{n-1}) = p, \; P(l_n = \tfrac{1-q}{1-p} | \mathcal{F}_{n-1}) = 1-p$$

Generalization:

$\vec{p} = [p_1\, p_2\, \cdots\, p_s]$, $\vec{q} = [q_1\, q_2\, \cdots\, q_s]$

$p_i, q_i \geq 0$ and $\sum p_i = \sum q_i = 1$

T_i, $i = 1, \ldots, s$, permutation matrices

$$P = \begin{bmatrix} \vec{p}T_1 \\ \vec{p}T_2 \\ \vdots \\ \vec{p}T_s \end{bmatrix} \qquad Q = \begin{bmatrix} \vec{q}T_1 \\ \vec{q}T_2 \\ \vdots \\ \vec{q}T_s \end{bmatrix}$$

Cyclic case

$$\begin{bmatrix} p_1 & p_2 & p_3 & 0 & \cdots & 0 \\ 0 & p_1 & p_2 & p_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ p_2 & p_3 & 0 & 0 & \cdots & p_1 \end{bmatrix}$$

T_i can be time varying.

AR Processes

$H_0:\quad \xi_n = w_n$, w_n: i.i.d. uniform on [-1 1]

$H_1:\quad \xi_n = \alpha\xi_{n-1} + w_n$,

w_n: i.i.d. $f_1(w)$ on $[-(1-\alpha)\,(1-\alpha)]$

$$\begin{aligned} P_n(l_n \leq x|\mathcal{F}_{n-1}) &= 0.5\nu\{\xi_n : 2f_1(\xi_n - \alpha\xi_{n-1}) \leq x\} \\ &= 0.5\nu\{w : 2f_1(w) \leq x\} \end{aligned}$$

Random Walk on a Circle

$H_0 : \{\xi_n\}$ i.i.d. uniform on unit circle

$H_1 : \xi_n = g(\xi_{n-1} + w_n)$, w_n i.i.d. $f_1(w)$

$g(\xi) = \xi - 2k\pi$ for $2k\pi \leq \xi < 2(k+1)\pi$

The transition density under H_1

$$h(\xi_n|\xi_{n-1}) = \sum_{k=-\infty}^{\infty} f_1(\xi_n - \xi_{n-1} + 2k\pi)$$

therefore

$$l_n = 2\pi \sum_{k=-\infty}^{\infty} f_1(\xi_n - \xi_{n-1} + 2k\pi)$$

$$P_n(l_n \leq x|\mathcal{F}_{n-1}) =$$
$$(2\pi)^{-1}\nu\{w : 2\pi \sum_{k=-\infty}^{\infty} f_1(w + 2k\pi) \leq x\}$$

Extension of Wald's First Identity to Markov Processes

Let $X_1, X_2, \ldots,$ i.i.d. and $S_n = \sum_{k=1}^{n} X_k$.

Simplest form: If $E[|X_1|] < \infty$ and N stopping time with $E[N] < \infty$ then

$$\boxed{E[S_N] = E[\sum_{n=1}^{N} X_n] = E[X_1]E[N]}$$

If $E[X_1] = 0$ then $E[S_N] = 0$.

Generalizations consider $E[X_1] = 0$ and relax $E[N] < \infty$.

If $E[|X_1|^\alpha] < \infty$ and $E[N^{1/\alpha}] < \infty$, $1 \leq \alpha \leq 2$, then $E[S_N] = 0$.

The Markov Case

Let $\{\xi_n\}$ a homogeneous Markov process and $\theta(\xi)$ a scalar nonlinearity. Consider $X_n = \theta(\xi_n)$ and $S_n = \sum_{k=1}^{n} \theta(\xi_k)$

$$E[\sum_{n=1}^{N} \theta(\xi_n)] = ?$$

A first result

$$E[S_N] = \mu'(0)E[N] - E[r'(\xi_N, 0)] + E[r'(\xi_0, 0)]$$

$\mu(s), r(\xi, s)$ are solutions to the eigenvalue problem

$$\begin{aligned} y(\xi) &= E[e^{s\theta(\xi_1)}x(\xi_1)|\xi_0 = \xi] \\ e^{\mu(s)}r(\xi, s) &= E[e^{s\theta(\xi_1)}r(\xi_1, s)|\xi_0 = \xi] \end{aligned}$$

Proposed Extension:

$$\boxed{E[S_N] = \lim_{n\to\infty} E[\theta(\xi_n)]E[N] + E[\omega(\xi_0)] - E[\omega(\xi_N)]}$$

where $\omega(\xi)$ satisfies a **Poisson Integral Equation** that has closed form solution for several interesting cases.

Requirements

1. Existence of invariant measure π.
2. Class of functions $\theta(\xi)$: $E_\pi[|\theta(\xi)|] < \infty$.
3. Type of ergodicity $E[\theta(\xi_n)] \to E_\pi[\theta(\xi)]$.

Background

Meyn & Tweedie: Markov Chains and Stochastic Stability.

Theorem (Meyn and Tweedie): Let $\{\xi_n\}$ irreducible and aperiodic then the following two conditions are equivalent:

i) There exists function $V(\xi) \geq 1$, a proper set C and constants $0 \leq \lambda < 1$, $b < \infty$ such that the following *Drift Condition* is satisfied

$$E[V(\xi_1)|\xi_0 = \xi] \leq \lambda V + b\mathbb{1}_C$$

$V(\xi)$ is called *Drift Function.*

ii) There exists probability measure π, function $V(\xi) \geq 1$ and constants $0 \leq \rho < 1$, $R < \infty$ such that

$$\sup_{|g| \leq V} |E[g(\xi_n)|\xi_0 = \xi] - \pi g| \leq \rho^n R V(\xi)$$

Denote $P^n g = E[g(\xi_n)|\xi_0 = \xi]$

The drift condition can be written as

$$\boxed{PV \leq \lambda V + b\mathbb{1}_C}$$

Define space of function $\mathcal{L}_V^\infty$ to be all measurable functions $g(\xi)$ such that

$$\sup_\xi \frac{|g(\xi)|}{V(\xi)} < \infty$$

Define also a norm $\|g\|_V$ in $\mathcal{L}_V^\infty$ to be

$$\|g\|_V = \sup_\xi \frac{|g(\xi)|}{V(\xi)}$$

then $\mathcal{L}_V^\infty$ is Banach. Furthermore for $g \in \mathcal{L}_V^\infty$ we have, due to Theorem 1

$$\boxed{|P^n g - \pi g| \leq \rho^n R \|g\|_V V(\xi)}$$

Lemma: Let $\theta(\xi) \in \mathcal{L}_V^\infty$ consider the Poisson Integral Equation with respect to the unknown $\omega(\xi)$

$$P\omega = \omega - (P\theta - \pi\theta), \quad \pi\omega = 0$$

then the unique solution in $\mathcal{L}_V^\infty$ is

$$\omega = \sum_{n=1}^{\infty} (P^n\theta - \pi\theta)$$

Theorem: Let $E[V(\xi_0)] < \infty$ then for any $\theta(\xi) \in \mathcal{L}_V^\infty$ we have

$$\begin{aligned} E[S_N] &= E[\sum_{n=1}^{N} \theta(\xi_n)] \\ &= (\pi\theta)E[N] + E[\omega(\xi_0)] - E[\omega(\xi_N)] \end{aligned}$$

$$\lim_{E[N]\to\infty} \frac{E[\omega(\xi_0)] - E[\omega(\xi_N)]}{E[N]} = 0$$

Examples

Finite State Markov Chains

Let ξ_n have K states and P denote the transition probability matrix.

P has a unit eigenvalue, if this eigenvalue is simple and all other eigenvalues have magnitude strictly less than unity then the chain is irreducible and aperiodic and an invariant measure π exists being the left eigenvector to the unit eigenvalue of P, i.e. $\pi^t P = \pi^t$ and $[1 \cdots 1]\pi = 1$.

Any function $\theta(\xi)$ can be regarded as a vector θ of length K and its expectation under the invariant measure is simply $\pi^t\theta$.

The Poisson Equation and the constraint takes here the form

$$\begin{aligned}(P-I)\omega &= -(P-J\pi^t)\theta\\ \pi^t\omega &= 0\end{aligned}$$

where I is the identity matrix and $J=[1\cdots 1]^t$.

If the null space of P is nontrivial then we can find vectors θ with corresponding $\omega=0$.

Finite Dependence

Consider $\{\zeta_n\}_{n=-m+1}^{\infty}$ i.i.d. with probability measure μ. Define $\xi_n = (\zeta_n, \zeta_{n-1}, \ldots, \zeta_{n-m+1})$. For simplicity consider $m = 2$, i.e. $\xi_n = (\zeta_n, \zeta_{n-1})$ and we are interested in $\theta(\zeta_n, \zeta_{n-1})$.

The invariant measure exists and it is equal to $\pi = \mu \times \mu$. Furthermore one can show that the process is irreducible and aperiodic. In fact we can see that $P^n = \pi$ for $n \geq 2$. This means that the solution to the Poisson Equation is

$$\omega = \sum_{n=1}^{\infty} P^n \theta - \pi\theta = P\theta - \pi\theta$$

or

$$\omega(\zeta) = E[\theta(\zeta_1, \zeta_0) | \zeta_0 = \zeta] - E[\theta(\zeta_1, \zeta_0)]$$

Generalized Wald's identity takes the form

$$E[\sum_{n=1}^{N} \theta(\zeta_n, \zeta_{n-1})] = E[\theta(\zeta_1, \zeta_0)]E[N] + E[\omega(\zeta_0)] - E[\omega(\zeta_N)]$$

where

$$\omega(\zeta) = E[\theta(\zeta_1, \zeta_0)|\zeta_0 = \zeta] - E[\theta(\zeta_1, \zeta_0)]$$

Finding $\theta(\xi)$ functions for which $\omega(\xi) = 0$ is easy. Let $g(\zeta_1, \zeta_0)$ be such that $\pi|g| < \infty$ then if

$$\theta(\zeta_1, \zeta_0) = g(\zeta_1, \zeta_0) - E[g(\zeta_1, \zeta_0)|\zeta_0] + c$$

we have $\omega(\zeta) = 0$.

AR Processes

We consider the scalar case

$$\xi_n = \alpha\xi_{n-1} + w_n, \quad \{w_n\} \text{ i.i.d.}, |\alpha| < 1$$

Lemma: If w_n has an everywhere positive density then $\{\xi_n\}$ is irreducible and aperiodic.

1. If $E[|w_1|^p] < \infty$ then $V(\xi) = 1+|\xi|^p$ is a drift function.

2. If for $c > 0$ we have $E[e^{c|w_1|^p}] < \infty$ (true for $1 \leq p \leq 2$ when w_n is Gaussian) then there exists $\delta > 0$ such that $V(\xi) = e^{\delta|\xi|^p}$ is a drift function.

Finding closed form expressions is not easy here. Special case where this is possible:

Polynomials.

If we have available the moments $E[w_1^j],\ j = 0,\dots,p$, then we can define polynomials $s_j(\xi),\ j = 0,\dots,p$ such that

$$Ps_j = \alpha^j s_j$$

with the coefficient of the highest power equal to 1. If w_n zero mean Gaussian then s_j are the normalized Hermite polynomials.

Any polynomial $\theta(\xi)$ of degree $k \leq p$ can be written as

$$\theta(\xi) = \theta_0 + \theta_1 s_1(\xi) + \cdots + \theta_p s_p(\xi)$$

Because of the fact that $Ps_j = \alpha^j s_j$ we conclude that

$$P^n\theta(\xi) = \theta_0 + \theta_1\alpha^n s_1(\xi) + \cdots + \theta_p\alpha^{pn}s_p(\xi)$$

and $\pi\theta = \lim_{n\to\infty} P^n\theta = \theta_0$.

To find $\omega(\xi)$ we apply the series and we have that

$$\omega(\xi) = \sum_{j=1}^{p} \theta_j \frac{\alpha^j}{1-\alpha^j} s_j(\xi)$$