Performance evaluation of CUSUM tests

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Outline

- Change detection and the CUSUM test
- Evaluating performance for BM
- The CUSUM test for Poisson processes
  - CUSUM performance when the rate decreases
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- Simulations

Change detection and the CUSUM test

We are observing sequentially a process $\{\xi_t\}$ with the following statistics:

$$\begin{aligned} \xi_t &\sim \mathbb{P}_\infty && \text{for } 0 \leqslant t \leqslant \tau \\ &\sim \mathbb{P}_0 && \text{for } \tau < t \end{aligned}$$

Define the CUSUM process y_t as follows:

$$y_t = u_t - m_t$$

where

$$u_t = \log\Big(\frac{d\mathbb{P}_0}{d\mathbb{P}_\infty}(\mathcal{F}_t)\Big) \qquad m_t = \inf_{0 \leqslant s \leqslant t} u_s$$

and the CUSUM stopping time

$$T = \inf_t \{\ t\colon y_t \geqslant \nu\ \}$$

We are interested in finding expressions for:

$\mathbb{E}_0[\,T\,]$: the worst Detection Delay

$\mathbb{E}_\infty[\,T\,]$: the False Alarm Delay

The only case where closed form expressions are available is when detecting changes in the constant drift of a Brownian Motion. For a drift that changes from 0 to μ we have

$$\mathbb{E}_0[\,T\,] = \frac{2}{\mu^2}(\nu + e^{-\nu} - 1)$$

$$\mathbb{E}_\infty[\,T\,] = \frac{2}{\mu^2}(e^{\nu} - \nu - 1)$$

Evaluating performance for BM

Let $\{\xi_t\}$ be defined as

$$d\xi_t = \begin{cases} dw_t & 0 \leqslant t \leqslant \tau \\ \mu dt + dw_t & \tau < t \end{cases}$$

$\{w_t\}$ standard Brownian Motion.

Define the log-likelihood ratio u_t between the two measures

$$du_t = \mu d\xi_t - 0.5\mu^2 dt$$

and the CUSUM process y_t

$$y_t = u_t - m_t$$

where
$$m_t = \inf_{0 \leqslant s \leqslant t} u_s$$

The CUSUM stopping time: $T = \inf_t \{ t \colon y_t \geqslant \nu \}$

Let $f(y)$ be a twice continuously differentiable function, using standard Ito calculus, we can write

$$df(y_t) = f'(y_t)(du_t - dm_t) + 0.5\mu^2 f''(y_t)dt$$

Key Idea: Eliminate any contribution coming from m_t

$dm_t \neq 0$ only when $u_t = m_t$ or $y_t = u_t - m_t = 0$

If $p(y)$ continuous with $p(0) = 0$, then $p(y_t)dm_t = 0$

Therefore the differential

$$df(y_t) = f'(y_t)(du_t - dm_t) + 0.5\mu^2 f''(y_t)dt$$

by simply imposing the constraint $f'(0)=0$ becomes:

$$df(y_t) = f'(y_t)du_t + 0.5\mu^2 f''(y_t)dt$$

Under $\mathbb{P}_0$: $du_t = \mu d\xi_t - 0.5\mu^2 dt = \mu(\mu dt + dw_t) - 0.5\mu^2 dt$
$= 0.5\mu^2 dt + \mu dw_t$

$$\mathbb{E}_0[f(y_T) - f(y_0)] = \mathbb{E}_0[\int_0^T 0.5\mu^2\{f'(y_t) + f''(y_t)\}dt]$$
$$+ \mathbb{E}_0[\int_0^T \mu f'(y_t)dw_t] = 0$$

$$\mathbb{E}_0[f(y_T)\text{-}f(y_0)] = \mathbb{E}_0[\int_0^T 0.5\mu^2\{f'(y_t)+f''(y_t)\}dt\ ]$$

$$y_T = \nu, \quad y_0 = 0, \quad f'(0) = 0$$

If we require: $0.5\mu^2\{f'(y)+f''(y)\} = \text{-}1$

with $f(\nu) = 0,\ f'(0) = 0$

then $f(y) = \dfrac{2}{\mu^2}\{(\nu + e^{-\nu} - 1)\text{-}(y + e^{-y} - 1)\}$

Substituting in the top equation we obtain

$$\mathbb{E}_0[\,T\,] = f(0) = \frac{2}{\mu^2}(\nu + e^{-\nu} - 1)$$

We Summarize:

$$\mathbb{E}[f(y_T) - f(y_0)] = \mathbb{E}[\int_0^T \mathcal{A}f(y_t)dt\ ]$$

Key point is the elimination of the contribution of m_t through a suitable constraint (case dependent).

If m_t has continuous paths it is sufficient to set

$$f'(0) = 0.$$

furthermore

$$\mathcal{A}f(y) = -1$$

$$f(y) = 0 \text{ for } y \geqslant \nu$$

If we are able to solve this problem and find $f(y)$ then

$$\mathbb{E}[\ T\ ] = f(0)$$

CUSUM for Poisson

Let $\{\mathcal{N}_t\}$ be a homogeneous Poisson process with rate λ satisfying the following change model:

$$\lambda = \begin{cases} \lambda_\infty, & 0 \leqslant t \leqslant \tau \\ \lambda_0, & \tau < t \end{cases}$$

$$u_t = -(\lambda_0 - \lambda_\infty)t + \log(\lambda_0/\lambda_\infty)\mathcal{N}_t$$

$$m_t = \inf_{0 \leqslant s \leqslant t} u_s$$

$$y_t = u_t - m_t$$

Let us first obtain the expression

$$\mathbb{E}[f(y_T) - f(y_0)] = \mathbb{E}[\int_0^T \mathcal{A} f(y_t) dt]$$

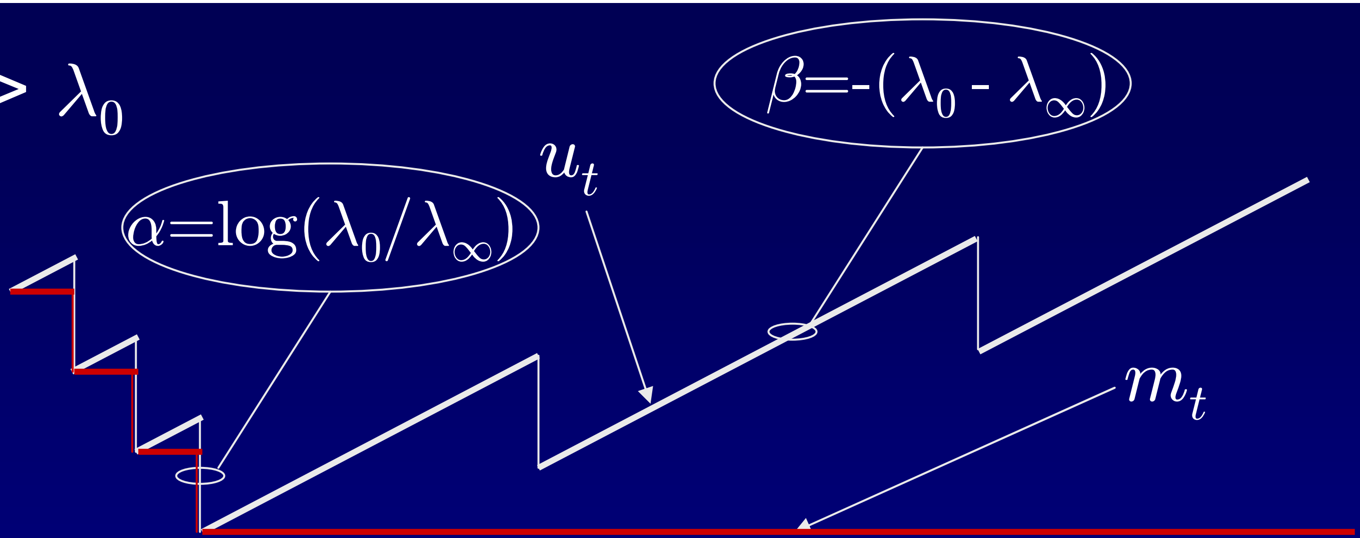
If $f(y)$ continuous with left and right derivatives, define $f'(y)$ to be left continuous, then

$$f(y_t) - f(y_0) = \int_{0+}^{t} f'(y_{s-}) dy_s^c + \sum_{Un \leqslant t} [f(y_{Un}) - f(y_{Un-})]$$

y_t^c: continuous part of y_t (between arrivals)

Un: arrival times

λ_∞ > λ_0



$$y_t = u_t - m_t$$

m_t has discontinuous paths (and the jumps are random)

$$f(y_t) - f(y_0) = \int_{0+}^{t} f'(y_{s\text{-}})dy_s^c + \sum_{Un \leqslant t} [f(y_{Un}) - f(y_{Un\text{-}})]$$

$$f(y_t) - f(y_0) = \int_{0+}^{t} \beta f'(y_{s\text{-}})dt +$$

$$\sum_{Un \leqslant t} [f(y_{Un\text{-}}+\alpha) - f(y_{Un\text{-}})] + \underbrace{[f(0) - f((y_{Un\text{-}}+\alpha)^-)]}_{\text{due to } m_t}$$

$$f(y_t) - f(y_0) = \int_{0+}^{t} \beta f'(y_{s-})dt + \sum_{Un \leqslant t} [f(y_{Un-}+\alpha) - f(y_{Un-})] + [f(0) - f((y_{Un-}+\alpha)^-)]$$

$$f(y_t) - f(y_0) =$$

due to m_t

$$\int_{0+}^{t} \beta f'(y_{s-})dt + [f(y_{s-}+\alpha) - f(y_{s-}) + \underbrace{f(0) - f((y_{s-}+\alpha)^-)}_{\text{due to } m_t}]d\mathcal{N}_s$$

To eliminate the contribution of m_t we impose the constraint:

$f(x) = f(0)$ for $x \leqslant 0$

$$f(y_t) - f(y_0) = \int_{0+}^{t} \beta f'(y_{s-})dt + [f(y_{s-}+\alpha) - f(y_{s-})]d\mathcal{N}_s$$

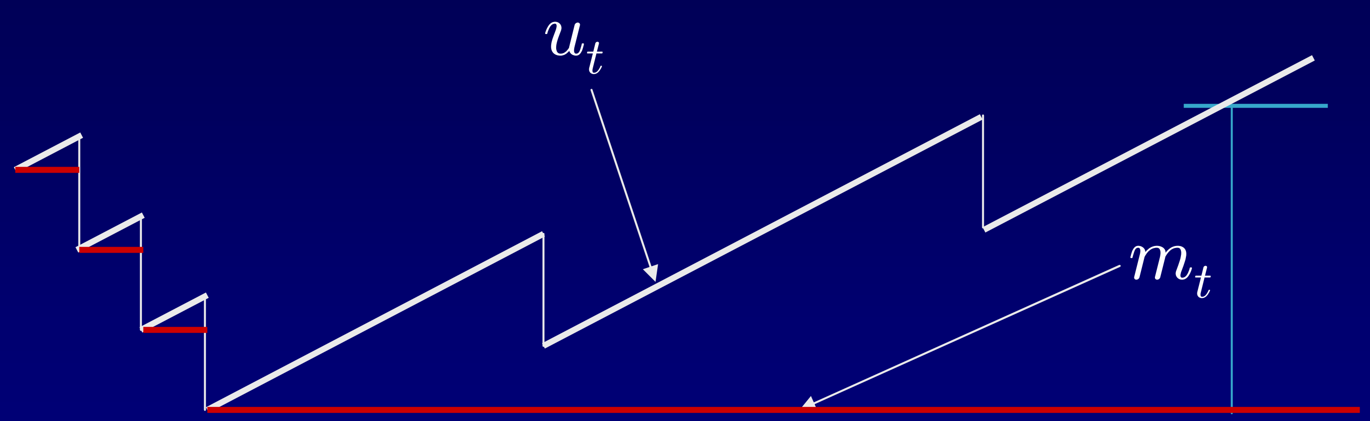
$$f(y_T) - f(y_0) = \int_{0+}^{T} \beta f'(y_{s-})dt + [f(y_{s-}+\alpha) - f(y_{s-})]d\mathcal{N}_s$$

$$d\mathcal{N}_t = \lambda_i dt + (d\mathcal{N}_t - \lambda_i dt),\ i = 0,\ \infty$$

$$\mathbb{E}_i[f(y_T) - f(y_0)] =$$

$$\mathbb{E}_i[\int_{0+}^{T} \{ \beta f'(y_{s-}) + \lambda_i [f(y_{s-}+\alpha) - f(y_{s-})] \}dt$$

$$\mathcal{A}f(y) = \beta f'(y) + \lambda_i [f(y+\alpha) - f(y)]$$



CUSUM stops between arrivals and **hits exactly** the threshold therefore we can impose the constraint

$$f(\nu) = 0$$

$$\mathcal{A}f(y) = \beta f'(y) + \lambda_i[f(y+\alpha) - f(y)] = -1,\ 0 \leqslant y \leqslant \nu$$

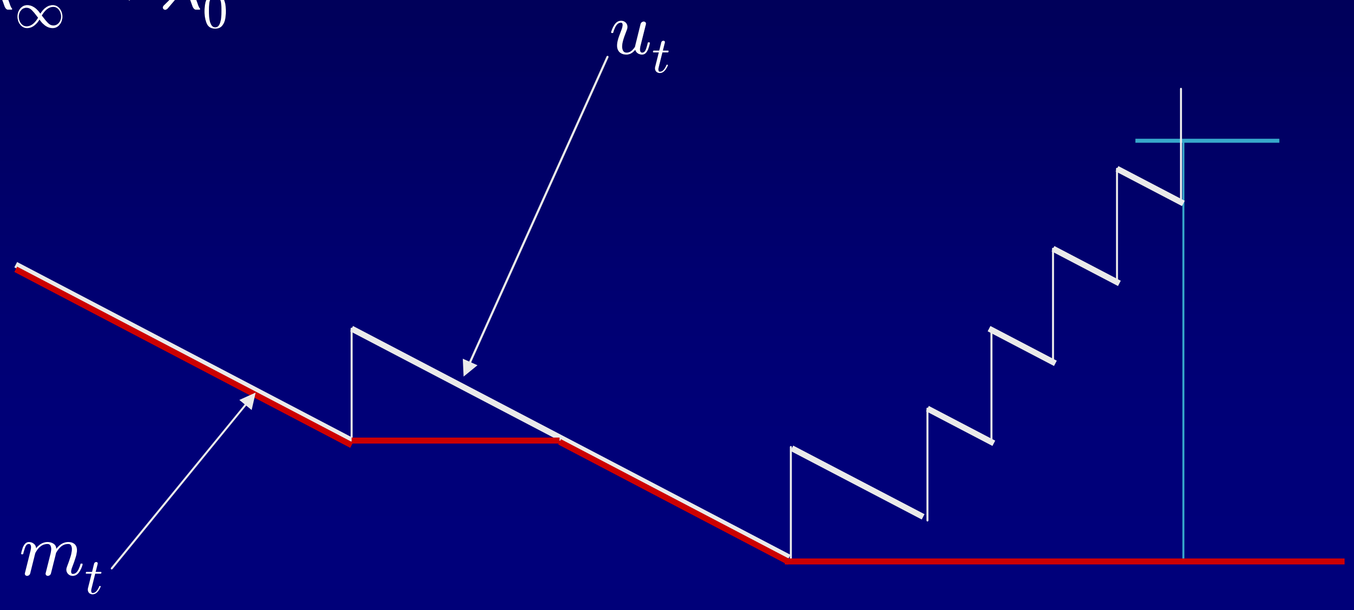
$$f(y) = f(0) \text{ for } y \leqslant 0$$

$$f(\nu) = 0$$

$$\mathbb{E}_i[\,T\,] =$$

$$\frac{1}{\lambda_i}\sum_{n=0}^{[\frac{\nu}{\beta}]}\left\{e^{-\lambda_i(\nu-n\beta)/\alpha}\left(\sum_{k=0}^{n}\frac{\left(\frac{\lambda_i(\nu-n\beta)}{\alpha}\right)^k}{k!}\right)-1\right\}$$

$\lambda_\infty < \lambda_0$



m_t has continuous paths therefore $f'(0) = 0$ eliminates m_t.

CUSUM stops at arrivals and it **overshoots** the threshold

$$\mathcal{A}f(y) = \beta f'(y) + \lambda_i[f(y+\alpha) - f(y)] = -1,\ 0 \leqslant y \leqslant \nu$$

$$f(y) = 0 \text{ for } y \geqslant \nu$$

$$f'(0) = 0$$

$$\mathbb{E}_i[\,T\,] =$$

$$\frac{1}{\lambda_i}\sum_{n=0}^{[\frac{\nu}{|\beta|}]}\left\{1 - e^{\lambda_i(\nu+n\beta)/\alpha}\sum_{k=0}^{n}\frac{\left(-\frac{\lambda_i(\nu+n\beta)}{\alpha}\right)^k}{k!}\right\}$$

$$+\frac{p_i}{\lambda_i}\sum_{n=0}^{[\frac{\nu}{|\beta|}]} e^{\lambda_i(\nu+n\beta)/\alpha}\frac{\left(-\frac{\lambda_i(\nu+n\beta)}{\alpha}\right)^n}{n!}$$

where p_i is defined as

$$p_i = \frac{A_i}{A_i - B_i}$$

with

$$A_i = \sum_{n=0}^{[\frac{\nu}{|\beta|}]} e^{\lambda_i(\nu+n\beta)/\alpha} \frac{\left(-\frac{\lambda_i(\nu+n\beta)}{\alpha}\right)^n}{n!}$$

$$B_i = \sum_{n=1}^{[\frac{\nu}{|\beta|}]} e^{\lambda_i(\nu+n\beta)/\alpha} \frac{\left(-\frac{\lambda_i(\nu+n\beta)}{\alpha}\right)^{n-1}}{(n-1)!}$$

Simulations

Average over 10000 repetitions:

$$\lambda_\infty = 1, \;\; \lambda_0 = 2, \; \nu = 5.5$$

	Analysis	Simulation
$\mathbb{E}_0[\,T\,] =$	12.2885	12.2673
$\mathbb{E}_\infty[\,T\,] =$	981.9811	986.7159

$$\lambda_\infty = 2, \;\; \lambda_0 = 1, \; \nu = 5.5$$

	Analysis	Simulation
$\mathbb{E}_0[\,T\,] =$	15.3832	15.3605
$\mathbb{E}_\infty[\,T\,] =$	779.9669	771.1219

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