Detecting changes in the rate of a Poisson process

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Outline

- Overview of the change detection problem
- CUSUM test and Lorden's criterion
- The Poisson disorder problem
  - CUSUM average run length for Poisson processes
  - CUSUM optimality in the sense of Lorden

Change detection - Overview

Available sequentially an observation process $\{\xi_t\}$ with the following statistics:

$$\xi_t \sim \mathbb{P}_\infty \quad \text{for } 0 \leqslant t \leqslant \tau$$
$$\sim \mathbb{P}_0 \quad \text{for } \tau < t$$

Detect change **as soon as possible**

- Change time τ :
  - Random with known prior (Bayesian)
  - Deterministic but unknown (Non-Bayesian)
- Known statistics $\mathbb{P}_\infty$, $\mathbb{P}_0$.

We are interested in **sequential schemes**.

With every new observation the test must decide

- **Stop** and issue an alarm
- Continue sampling

Decision at time t uses available information

$$\mathcal{F}_t = \sigma\{\xi_s : 0 \leqslant s \leqslant t\}.$$

up to time t.

Sequential test → stopping time T adapted to the filtration $\{\mathcal{F}_t\}$.

$\mathbb{P}_\tau$: the probability measure induced, when change takes place at time τ

$\mathbb{E}_\tau[.]$: the corresponding expectation

$\mathbb{P}_\infty$: all data under nominal regime

$\mathbb{P}_0$: all data under alternative regime

Parameters to be considered

- The detection delay T - τ
- Frequency of false alarms

Bayesian approach (Shiryayev 1978)

Change time τ random with exponential prior.

$$J(T) = c\,\mathbb{E}[\,(T - \tau)^+\,] + \mathbb{P}[\,T < \tau\,]$$

Optimization problem: $\inf_T J(T)$

$$\pi_t = \mathbb{P}[\,\tau \leqslant t \mid \mathcal{F}_t];\qquad T_{\mathrm{S}} = \inf_t\,\{\,t\colon \pi_t \geqslant \nu\,\}$$

- <u>Discrete time</u>: i.i.d. observations (Shiryayev 1978, Poor 1998)
- <u>Continuous time</u>: Brownian Motion (Shiryayev 1978, Beibel 2000, Karatzas 2003)

Non-Bayesian setup (Pollak 1985)

The change time τ is deterministic & unknown.

$$J(T) = \sup_{\tau} \mathbb{E}_{\tau}[\,(T - \tau) \mid T > \tau\,]$$

Optimization problem: $\inf_T J(T)$

subject to: $\mathbb{E}_{\infty}[\,T\,] \geqslant \gamma$

<u>Discrete time</u>: i.i.d. detect change in the pdf from $f_{\infty}(\xi)$ to $f_0(\xi)$. Roberts (1966) proposed

$$S_t = (S_{t-1} + 1)\,\frac{f_0(\xi_t)}{f_{\infty}(\xi_t)}$$

$$T_{\mathrm{SRP}} = \inf_t \{\, t\colon S_t \geqslant \nu \,\}$$

(Mei 2006)

CUSUM test and Lorden's criterion

Discrete time, i.i.d. observations.
Pdf before and after the change: $f_\infty(\xi_n)$, $f_0(\xi_n)$

Since change time τ is unknown

$$\sup_{0\leqslant\tau\leqslant t}\sum_{n=\tau+1}^{t}\log\left(\frac{f_0(\xi_n)}{f_\infty(\xi_n)}\right)\geqslant\nu$$

$$\sum_{n=1}^{t}\log\left(\frac{f_0(\xi_n)}{f_\infty(\xi_n)}\right) - \inf_{0\leqslant\tau\leqslant t}\sum_{n=1}^{\tau}\log\left(\frac{f_0(\xi_n)}{f_\infty(\xi_n)}\right)\geqslant\nu$$

$$u_t \quad - \quad m_t \quad \geqslant \nu$$

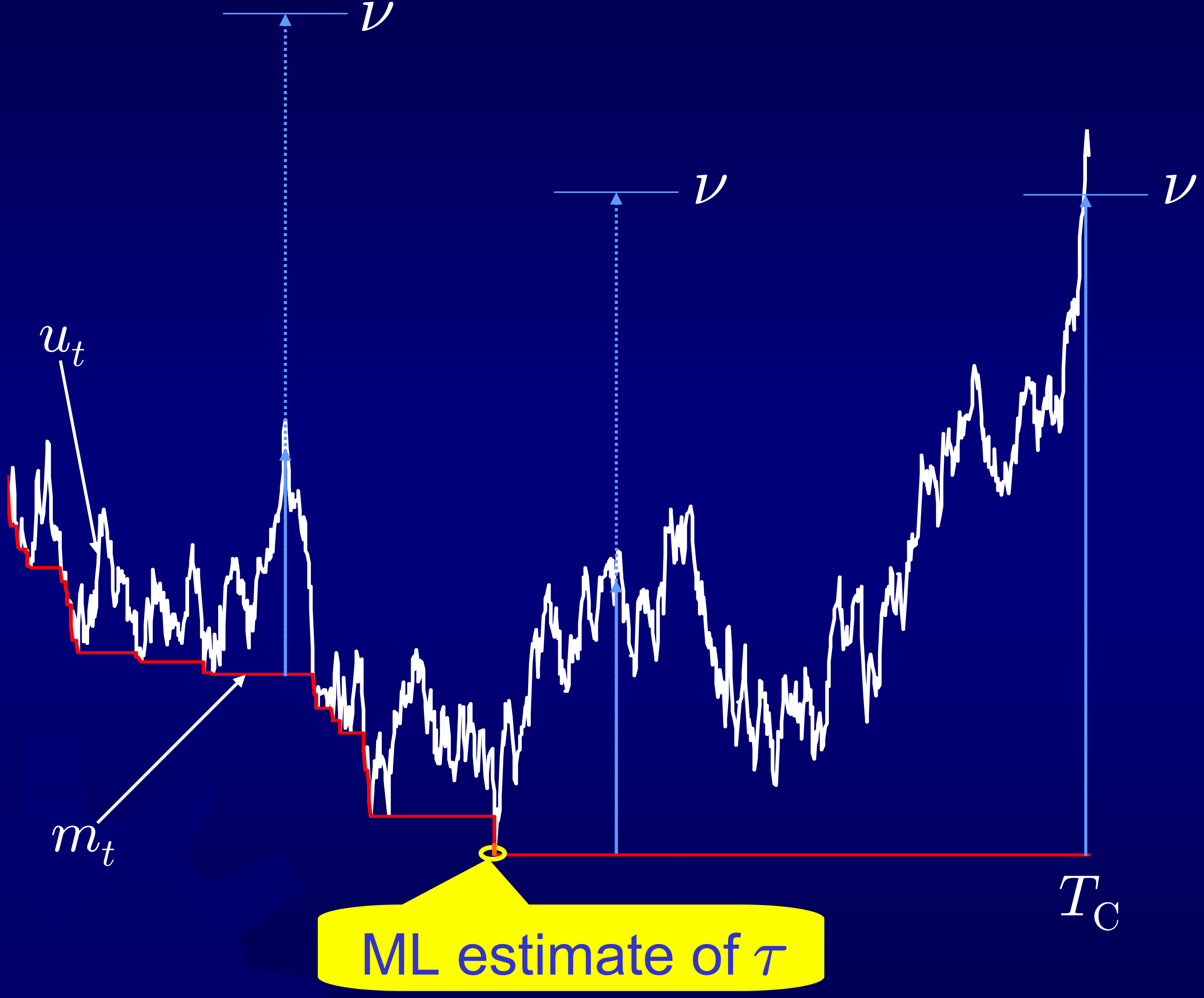
$$u_t = \log\left(\frac{d\mathbb{P}_0}{d\mathbb{P}_\infty}(\mathcal{F}_t) \right)$$

$$m_t = \inf_{0 \leqslant s \leqslant t} u_s$$

CUSUM process: $y_t = u_t - m_t \geqslant 0$

The CUSUM stopping time (Page 1954):

$$T_{\mathrm{C}} = \inf_t \{ t \colon y_t \geqslant \nu \}$$


ν
ν
ν
u_t
m_t
T_C
ML estimate of τ

Non-Bayesian setup (Lorden 1971).

Change time τ is deterministic and unknown.

$$J(T) = \sup_\tau \operatorname{essup} \mathbb{E}_\tau[\,(T - \tau)^+ \mid \mathcal{F}_\tau\,]$$

Optimization problem: $\inf_T J(T)$

subject to: $\mathbb{E}_\infty[\,T\,] \geqslant \gamma$

- Discrete time: i.i.d. observations (Moustakides 1986, Ritov 1990, Poor 1998)
- Continuous time: BM (Shiryayev 1996, Beibel 1996); Ito processes (Moustakides 2004)

The Poisson disorder problem

Let $\{\mathcal{N}_t\}$ homogeneous Poisson, with rate λ satisfying:

$$\lambda = \begin{cases} \lambda_\infty , & 0 \leqslant t \leqslant \tau \\ \lambda_0 , & \tau < t \end{cases}$$

Bayesian Approach

- Linear delay: Galchuk & Rozovski 1971, Davis 1976, Peskir & Shiryayev 2002.
- Exponential delay: Bayraktar & Dayanik 2003, B & D & Karatzas 2004 and 2005 (adaptive)

CUSUM & average run length

$$u_t = (\lambda_\infty - \lambda_0)t + \log(\lambda_0/\lambda_\infty)\mathcal{N}_t$$

$$m_t = \inf_{0\leqslant s\leqslant t} u_s$$

$$y_t = u_t - m_t$$

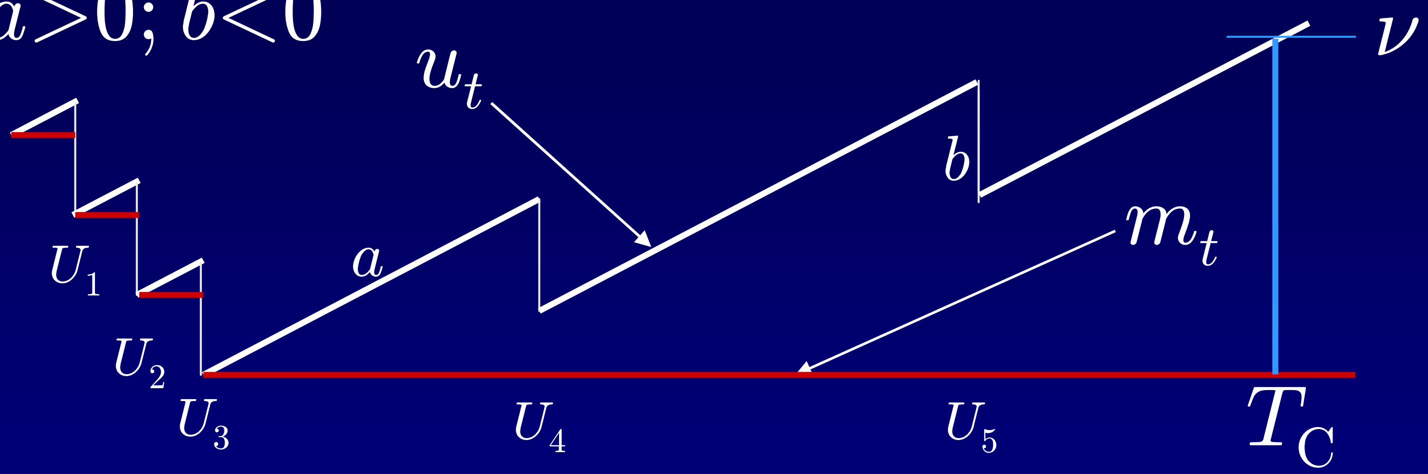
$$T_{\mathrm{C}} = \inf_t \{ t\colon y_t \geqslant \nu \}$$

We are interested in computing $\mathbb{E}[\ T_{\mathrm{C}}\ ]$ when $\mathcal{N}_t$ is Poisson with rate λ.

Existing formula (Taylor 1975) for:

$u_t = at + b\mathcal{W}_t$; $\mathcal{W}_t$ is standard Wiener

$a>0;\ b<0$



Find $f(y)$ such that $f(y_0)=\mathbb{E}[T_{\mathrm{C}}]$.

Study the paths of $f(y_t)$

$$
\begin{aligned}
f(y_t) - f(y_0) &= \int_0^t f'(y_{s-})a\,ds \\
&\quad + \int_0^t \left[f(y_{s-}+b) - f(y_{s-})\right] d\mathcal{N}_s
\end{aligned}
$$

$$f(y)=f(0);\ y\leqslant 0$$

$$f(y_{T_C}) - f(y_0) = \int_0^{T_C} f'(y_{t-})a\,dt + \int_0^{T_C} [f(y_{t-}+b) - f(y_{t-})]\,d\mathcal{N}_t$$

$$\mathbb{E}[f(y_{T_C})] - f(y_0) = \mathbb{E}\left[\int_0^{T_C}\{f'(y_{t-})a + \lambda[f(y_{t-}+b) - f(y_{t-})]\}dt\right]$$

$$af'(y) + \lambda[f(y+b) - f(y)] = -1;\ \ y \in [0,\nu)$$

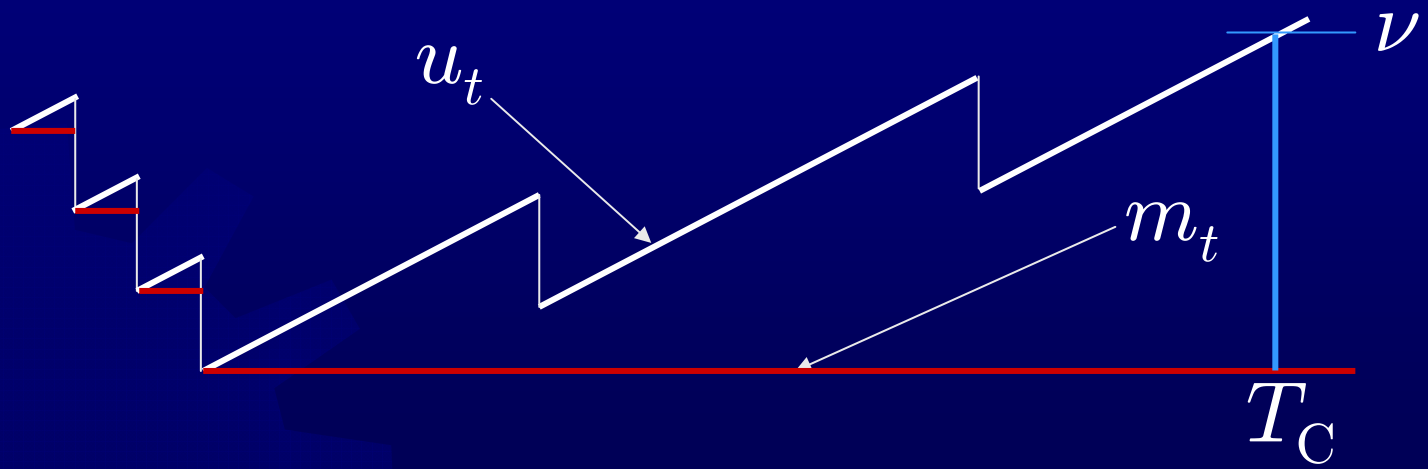
$$f(y_0) - \mathbb{E}[f(y_{T_C})] = \mathbb{E}[T_C]$$

We end up with a DDE and the following boundary conditions:

$$a f'(y) + \lambda[f(y+b) - f(y)] = -1;\ y \in [0,\nu)$$

$$f(y) = f(0) \text{ for } y \leqslant 0; \qquad f(\nu) = 0$$

$$f(y_0) = \mathbb{E}[T_{\mathrm{C}}]$$



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Existing formula (Taylor 1975) for:

$u_t = at + b\mathcal{W}_t$; $\mathcal{W}_t$ standard Wiener

$$\frac{b^2}{2} f''(y) + a f'(y) = -1; \; y \in [0,\nu]$$

$$f'(0) = 0; \qquad f(\nu) = 0$$

$$\mathbb{E}[T_C | y_0 = y] = f(y) = \frac{1}{a}\left\{\frac{b^2}{2a}\left(1 - e^{-\frac{2a}{b^2}\nu}\right) - \nu\right\} - \frac{1}{a}\left\{\frac{b^2}{2a}\left(1 - e^{-\frac{2a}{b^2}y}\right) - y\right\}$$

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$$af'(y) + \lambda[f(y+b) - f(y)] = -1;\ y\in[0,\nu)$$

$$f(y) = f(0) \text{ for } y \leqslant 0; \qquad f(\nu)=0$$

Because $b<0$ it is a forward DDE

$$\mathbb{E}[T_C|y_0 = y] = f(y) =$$

$$\frac{1}{\lambda}\sum_{n=0}^{[\frac{\nu}{|b|}]}\left\{e^{\frac{\lambda(\nu-n|b|)}{a}}\left(\sum_{k=0}^{n}\frac{\left(\frac{-\lambda(\nu-n|b|)}{a}\right)^k}{k!}\right)-1\right\}$$

$$-\frac{1}{\lambda}\sum_{n=0}^{[\frac{y}{|b|}]}\left\{e^{\frac{\lambda(y-n|b|)}{a}}\left(\sum_{k=0}^{n}\frac{\left(\frac{-\lambda(y-n|b|)}{a}\right)^k}{k!}\right)-1\right\}$$

$a<0;\quad b>0$

$$af'(y)+\lambda[f(y+b)-f(y)]=-1;\ y\in[0,\nu)$$

$f'(0)=0;$ $\qquad$ $f(y)=0$ for $y\geqslant\nu$

Backward DDE

$$\mathbb{E}[T_C|y_0=0]=f(0)=$$

$$\frac{1}{\lambda}\sum_{n=0}^{[\frac{\nu}{b}]}\left\{1-e^{\frac{\lambda(\nu-nb)}{|a|}}\sum_{k=0}^{n}\frac{\left(-\frac{\lambda(\nu-nb)}{|a|}\right)^k}{k!}\right\}$$

$$+\frac{p}{\lambda}\sum_{n=0}^{[\frac{\nu}{b}]}e^{\frac{\lambda(\nu-nb)}{|a|}}\frac{\left(-\frac{\lambda(\nu-nb)}{|a|}\right)^n}{n!}$$

where p is defined as

$$p = \frac{A}{A-B}$$

with

$$A = \sum_{n=0}^{[\frac{\nu}{b}]} e^{\frac{\lambda(\nu-nb)}{|a|}} \frac{\left(-\frac{\lambda(\nu-nb)}{|a|}\right)^n}{n!}$$

$$B = \sum_{n=1}^{[\frac{\nu}{b}]} e^{\frac{\lambda(\nu-nb)}{|a|}} \frac{\left(-\frac{\lambda(\nu-nb)}{|a|}\right)^{n-1}}{(n-1)!}$$

Average over 10000 repetitions:

$\lambda_\infty = 2,\ \ \lambda_0 = 1,\ (a{=}1,\ b{=}\text{-}\log 2),\ \nu = 5.5$

	Formula	Simulation
$\mathbb{E}_0[\ T_\mathrm{C}\ ]$	15.3832	15.3605
$\mathbb{E}_\infty[\ T_\mathrm{C}\ ]$	779.9669	771.1219

$\lambda_\infty = 1,\ \ \lambda_0 = 2,\ (a{=}\text{-}1,\ b{=}\log 2),\ \nu = 5.5$

	Formula	Simulation
$\mathbb{E}_0[\ T_\mathrm{C}\ ]$	12.2885	12.2673
$\mathbb{E}_\infty[\ T_\mathrm{C}\ ]$	981.9811	986.7159

Optimality of CUSUM

$$J(T) = \sup_\tau \operatorname{essup} \mathbb{E}_\tau[\ (T - \tau)^+ \mid \mathcal{F}_\tau\ ]$$

$\inf_T J(T);$ subject to: $\mathbb{E}_\infty[\ T\ ] \geqslant \gamma$

If T is such that

$$\mathbb{E}_\infty[\ T\ ] \geqslant \mathbb{E}_\infty[\ T_{\mathrm{C}}\ ] = \gamma$$

then

$$J(T) \geqslant J(T_{\mathrm{C}})$$

$$h(y) = \mathbb{E}_\infty[\, T_\mathrm{C} \mid y_0 = y]$$

$$g(y) = \mathbb{E}_0[\, T_\mathrm{C} \mid y_0 = y]$$

$$\mathrm{essup}\, \mathbb{E}_\tau[\,(T_\mathrm{C} - \tau)^+ \mid \mathcal{F}_\tau] = \sup_y g(y) = g(0)$$

T_C is an **equilizer rule** therefore

$$J(T_\mathrm{C}) = g(0)$$

For the false alarm we have

$$\mathbb{E}_\infty[\, T_\mathrm{C}\,] = h(0)$$

We would like to show:

If $\mathbb{E}_\infty[T] \geqslant \mathbb{E}_\infty[T_{\mathrm{C}}]$

then $J(T) \geqslant J(T_{\mathrm{C}})$

Lemma

$$J(T) \geq \frac{\mathbb{E}_\infty\left[\int_0^T e^{y_s} ds\right]}{\mathbb{E}_\infty\left[e^{y_T}\right]} = \mathcal{J}(T)$$

Sufficient:

If $\mathbb{E}_\infty[T] \geqslant h(0)$

then $\mathcal{J}(T) \geqslant g(0)$

$$\mathcal{J}(T) = \frac{\mathbb{E}_\infty\left[\int_0^T e^{y_s} ds\right]}{\mathbb{E}_\infty\left[e^{y_T}\right]} \geq g(0)$$

$$\mathbb{E}_\infty\left[\int_0^T e^{y_s} ds - g(0)e^{y_T}\right]$$
$$-\{\mathbb{E}_\infty[T] - h(0)\} \geq 0$$

$$\mathbb{E}_\infty\left[\int_0^T (e^{y_s} - 1) ds - g(0)e^{y_T} + h(0)\right] \geq 0$$

We will show that this is true for **any** T

Consider the function $f(y)$ defined as follows

$$f(y) = e^y \left[g(0) - g(y) \right] - \left[h(0) - h(y) \right]$$

then

$$\mathbb{E}_\infty[f(y_T)] = \mathbb{E}_\infty \left[\int_0^T (e^{y_s} - 1) 1_{\{y_s < \nu\}} ds \right]$$

$$\mathbb{E}_\infty \left[\int_0^T (e^{y_s} - 1) ds - g(0) e^{y_T} + h(0) \right] \overset{?}{\geqslant} 0$$

$$= \mathbb{E}_\infty \left[\int_0^T (e^{y_s} - 1) 1_{\{y_s \geq \nu\}} ds \right]$$

$$+ \mathbb{E}_\infty \left[h(y_T) - e^{y_T} g(y_T) \right] \geqslant 0$$

■

Conclusion

- We considered the Poisson disorder problem of detecting changes in the rate of a homogeneous Poisson process, in the sense of Lorden.
- We obtained closed form expressions for the average run length of the CUSUM stopping time.
- We used these formulas to prove optimality of the CUSUM test in the sense of Lorden.

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