Joint Detection & Estimation
Application to MIMO Radar

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Outline

- Joint Detection and Estimation
- Problem formulation
- Optimum combined strategy
- MIMO radar
- Application of optimum scheme
- Simulations

Joint Detection and Estimation

The problem: For a finite sequence of samples $X=[x_1,\ldots,x_n]$ we assume the following two hypotheses:

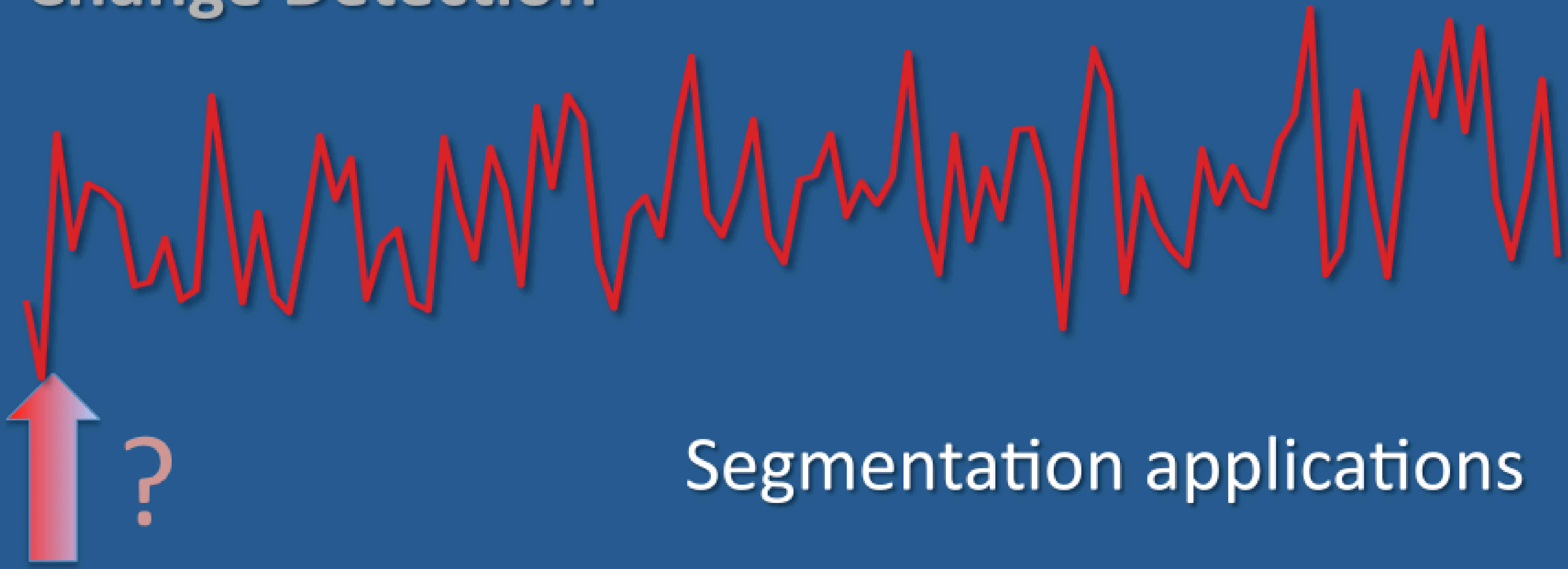
$$\mathrm{H}_0 : \ X \sim f_0(X)$$

$$\mathrm{H}_1 : \ X \sim f_1(X|\theta)$$

Detection: Given a data vector X, **decide** between the two hypotheses

Estimation: Every time there is a decision in favor of H_1 **estimate** θ

Change Detection



Segmentation applications

Detect if after some point in time there is a change in the statistical behavior of the data.

Every time we detect a change, we like to **estimate** the point of change.

Radar

If a plane enters the operational space of the radar, we would like to **detect** it.

Once a plane is detected we would also like to **estimate** its position, speed...

Detection problem (Neyman-Pearson)

We are given a data vector $X = [x_1, \ldots, x_n]$ for which we assume the following two hypotheses:

$$\mathsf{H}_0: \; X \sim f_0(X)$$
$$\mathsf{H}_1: \; X \sim f_1(X)$$

Use X to select $D = 0$ or 1

False alarm $\mathsf{P}(D = 1 | \mathsf{H}_0) \leq \alpha, \quad 0 < \alpha < 1$

Maximize $\mathsf{P}(D = 1 | \mathsf{H}_1)$

Minimize $\mathsf{P}(D = 0 | \mathsf{H}_1)$

$$\frac{f_1(X)}{f_0(X)} \mathop{\gtreqless}_{\mathsf{H}_0}^{\mathsf{H}_1} t$$

$$\mathsf{H}_0: \; X \sim f_0(X)$$

$$\mathsf{H}_1: \; X \sim f_1(X|\theta), \quad \pi(\theta)$$

False alarm $\mathsf{P}(D=1|\mathsf{H}_0) \leq \alpha, \quad 0 < \alpha < 1$

Minimize $\mathsf{P}(D=0|\mathsf{H}_1)$

$$\frac{\int f_1(X|\theta)\pi(\theta)d\theta}{f_0(X)} \underset{\mathsf{H}_0}{\overset{\mathsf{H}_1}{\gtreqless}} t$$

$$\mathsf{P}(D=0|\mathsf{H}_1) \geq \min \mathsf{P}(D=0|\mathsf{H}_1) = \beta_{\mathrm{NP}}(\alpha)$$

Estimation problem (Bayesian approach)

We are given a data vector $X=[x_1,\ldots,x_n]$ for which we assume the following model:

$$X \sim f_1(X|\theta), \quad \pi(\theta)$$

Use X to find an estimate $\hat{\theta}$ of θ

Minimize $\mathsf{E}_1\left[\|\hat{\theta}-\theta\|^2\right]$

$$\hat{\theta}_{\mathrm{MSE}} = \mathsf{E}_1[\theta|X] = \frac{\int \theta f_1(X|\theta)\pi(\theta)d\theta}{\int f_1(X|\theta)\pi(\theta)d\theta}$$

Formulation of the joint problem

$$\mathsf{H}_0: \quad X \sim f_0(X)$$

$$\mathsf{H}_1: \quad X \sim f_1(X|\theta), \quad \pi(\theta)$$

An obvious way to proceed is as follows:

For detection use:

$$\frac{\int f_1(X|\theta)\pi(\theta)d\theta}{f_0(X)} \underset{\mathsf{H}_0}{\overset{\mathsf{H}_1}{\gtreqless}} t$$

For estimation use:

$$\hat{\theta} = \frac{\int \theta f_1(X|\theta)\pi(\theta)d\theta}{\int f_1(X|\theta)\pi(\theta)d\theta}$$

There is the need for a formulation of the joint problem.

Caution: The theory we are going to present, in order to be applicable, we need the estimation of the parameters to be part of our goals.

Our theory is **NOT** applicable if we are interested only in detection and the random parameters are just nuisance terms.

Minimize $\mathsf{E}_1\left[\|\hat{\theta}-\theta\|^2 | D=1\right]$

$$\alpha \geq \mathsf{P}_0(D=1) \qquad \beta \geq \mathsf{P}_1(D=0) \geq \beta_{\mathrm{NP}}(\alpha)$$

Theorem: The optimum combined detection and estimation scheme is defined as follows:

$$\hat{\theta} = \hat{\theta}_{\mathrm{MSE}} = \frac{\int \theta f_1(X|\theta)\pi(\theta)d\theta}{\int f_1(X|\theta)\pi(\theta)d\theta}$$

$$\frac{\int f_1(X|\theta)\pi(\theta)d\theta}{f_0(X)}\left\{1-\lambda\sigma^2(X)\right\} \underset{\mathsf{H}_0}{\overset{\mathsf{H}_1}{\gtrless}} t$$

$$\sigma^2(X) = \mathsf{E}_1\left[\|\hat{\theta}-\theta\|^2 | X\right] = \frac{\int \|\theta-\hat{\theta}\|^2 f_1(X|\theta)\pi(\theta)d\theta}{\int f_1(X|\theta)\pi(\theta)d\theta}$$

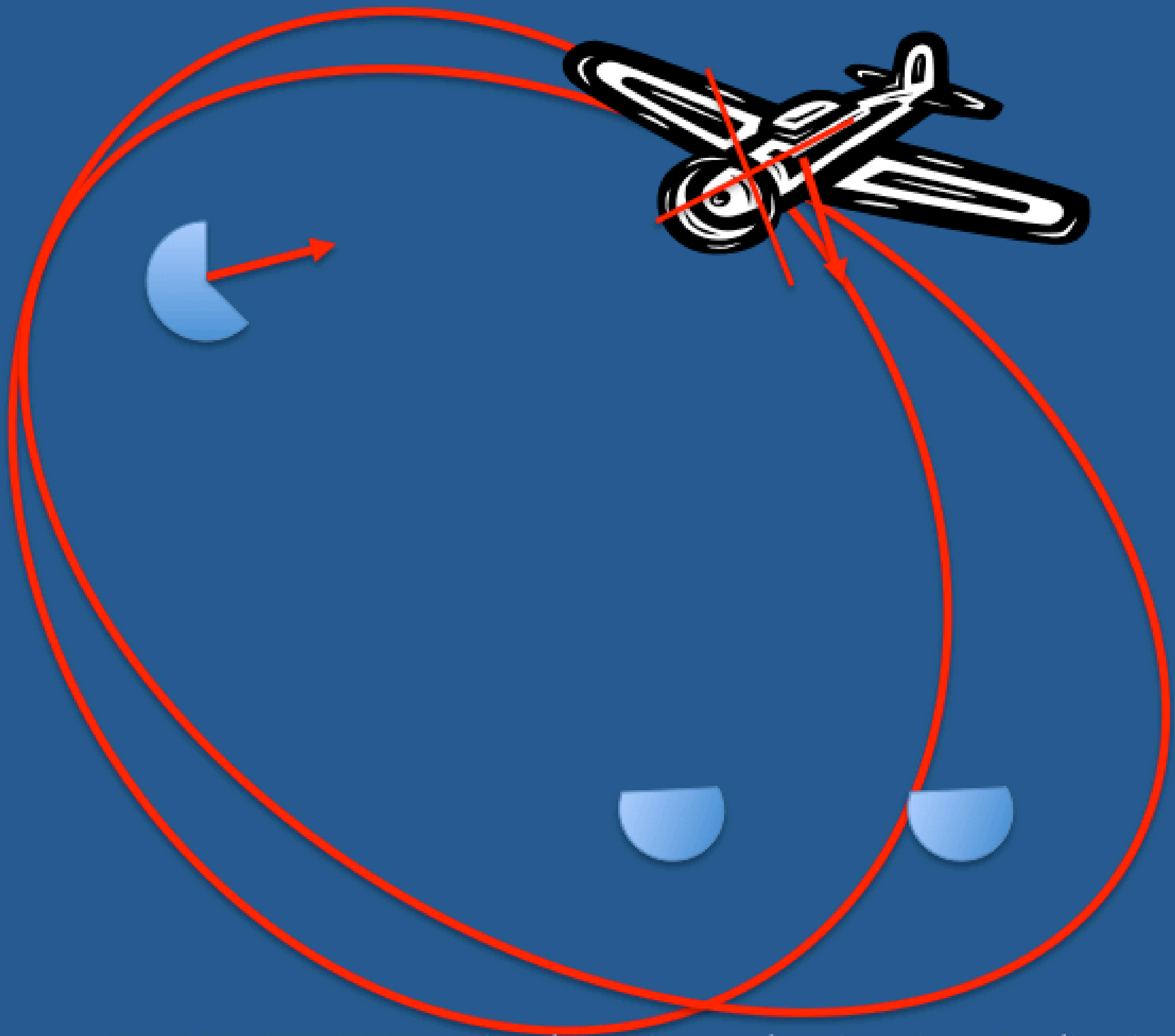
$$\hat{\theta} = \hat{\theta}_{\mathrm{MSE}} = \frac{\int \theta \mathcal{L}(X|\theta)\pi(\theta)d\theta}{\int \mathcal{L}(X|\theta)\pi(\theta)d\theta}$$

$$\int \mathcal{L}(X|\theta)\pi(\theta)d\theta \left\{1 - \lambda\sigma^2(X)\right\} \mathop{\gtreqless}_{\mathsf{H}_0}^{\mathsf{H}_1} t$$

$$\sigma^2(X) = \frac{\int \|\theta - \hat{\theta}\|^2 \mathcal{L}(X|\theta)\pi(\theta)d\theta}{\int \mathcal{L}(X|\theta)\pi(\theta)d\theta}$$

$$\mathcal{L}(X|\theta) = \frac{f_1(X|\theta)}{f_0(X)}$$

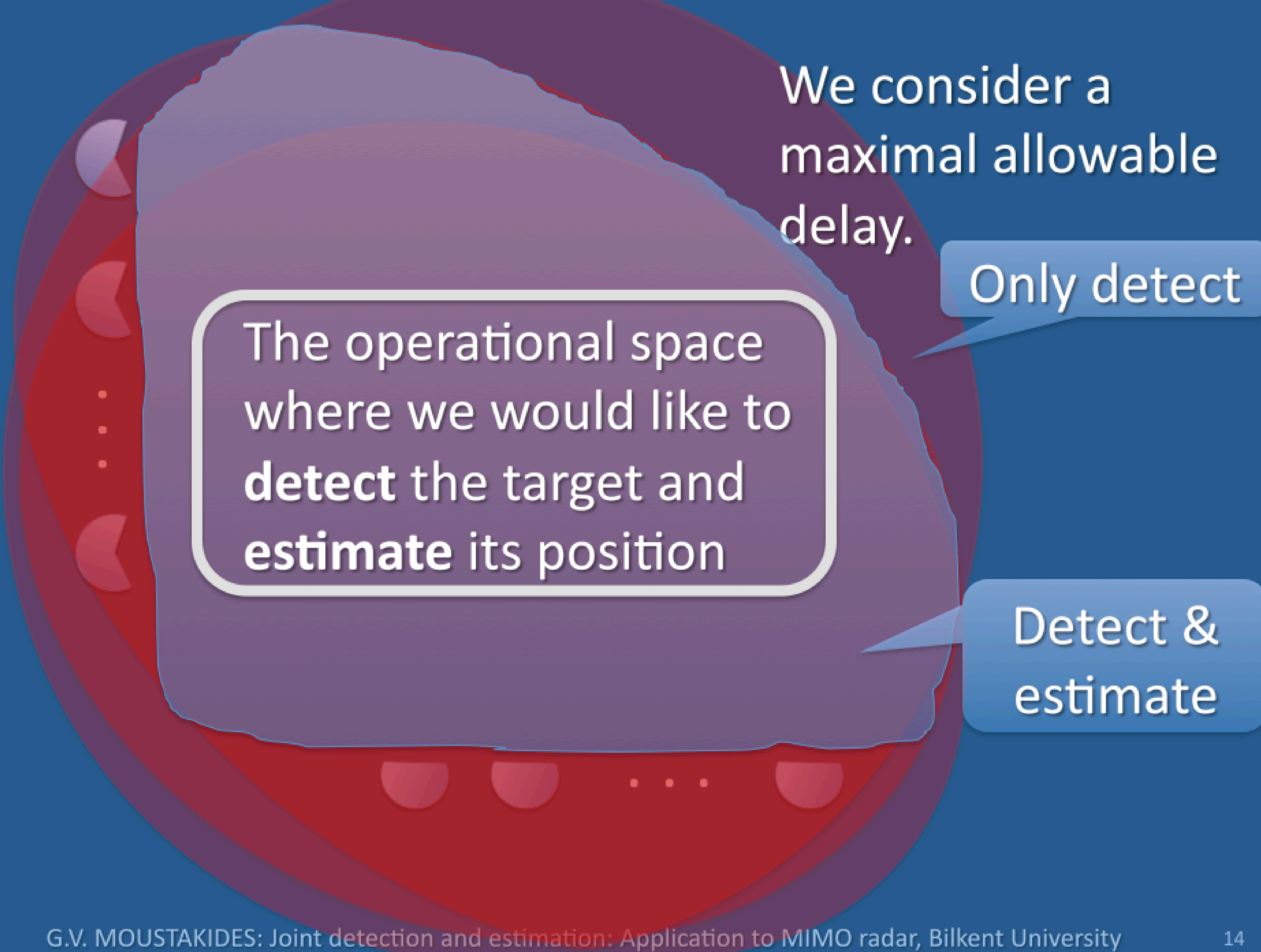
MIMO radar



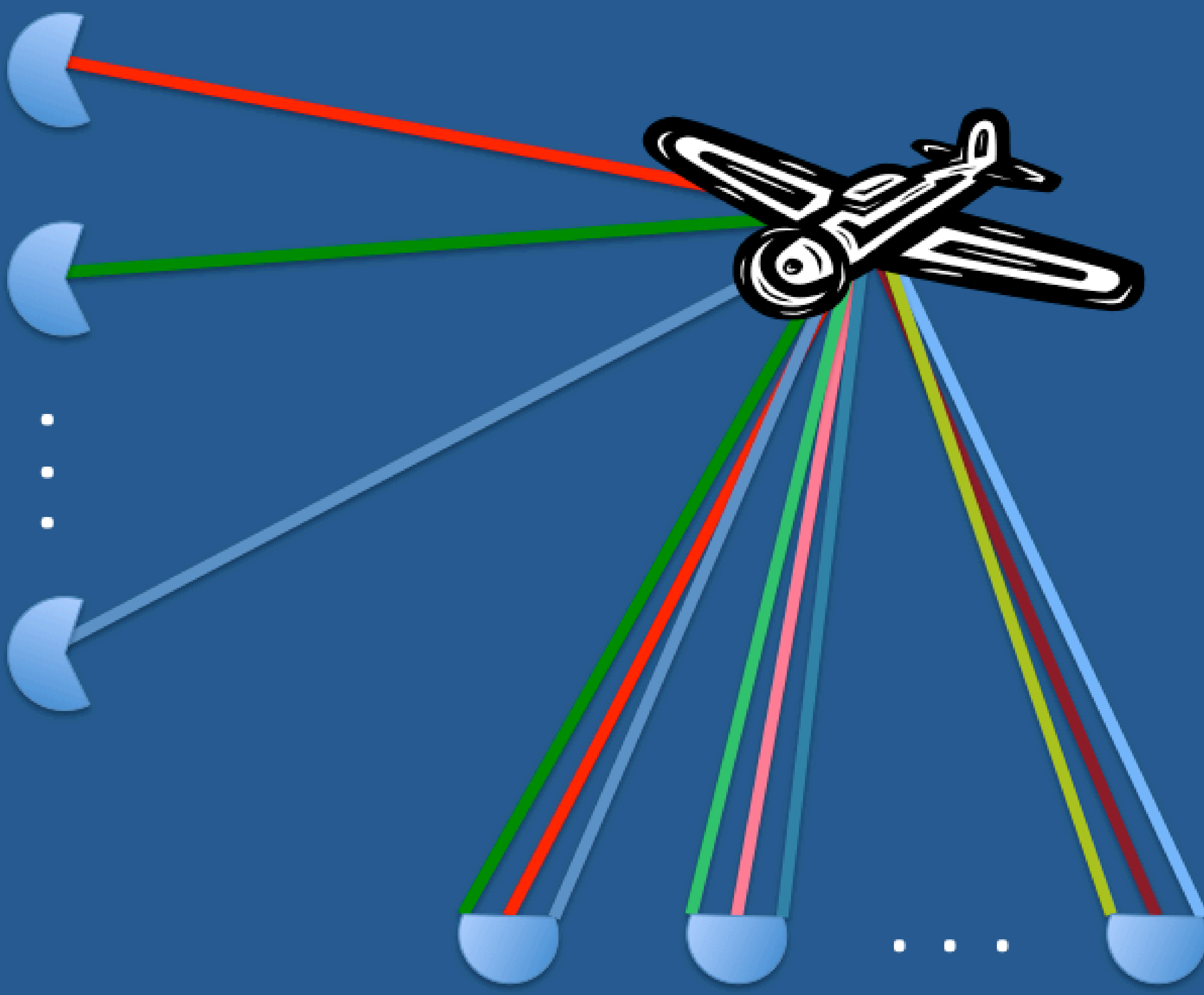
Measure the delay.

The delay is proportional to the distance traveled by the EM wave.

Defines an ellipse.


We consider a maximal allowable delay.
Only detect
The operational space where we would like to **detect** the target and **estimate** its position
Detect & estimate

Application of optimum scheme

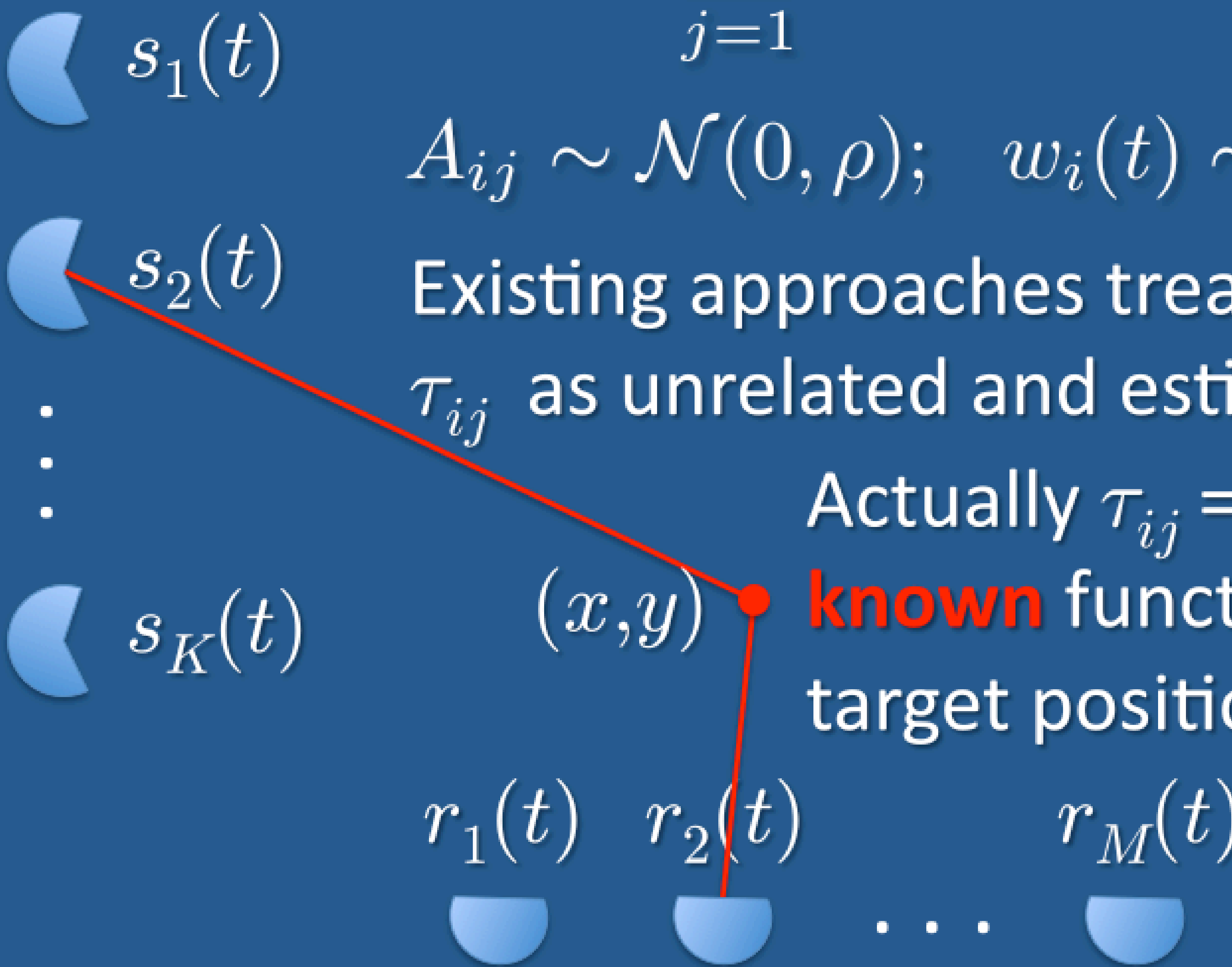


$$r_i(t) = \sum_{j=1}^{K} A_{ij} s_j(t - \tau_{ij}) + w_i(t)$$

$$A_{ij} \sim \mathcal{N}(0, \rho); \quad w_i(t) \sim \mathcal{N}(0, 1)$$

Existing approaches treat the delays τ_{ij} as unrelated and estimate them

Actually $\tau_{ij} = \tau_{ij}(x,y)$ are **known** functions of the target position



$$\mathsf{H}_0 : r_i(t) = w_i(t)$$

$$\mathsf{H}_1 : r_i(t) = \sum_{j=1}^{K} A_{ij} s_j(t - \tau_{ij}(x,y)) + w_i(t)$$

$$A_{ij} \sim \mathcal{N}(0,\rho); \quad w_i(t) \sim \mathcal{N}(0,1)$$

For the position (x,y) we assume **uniform** prior over the area where we **can detect and estimate.**

$$X \Longleftrightarrow [r_1(t),\ldots,r_M(t)] \qquad \theta \Longleftrightarrow (x,y)$$

$$\mathcal{L}(X|\theta) \Longleftrightarrow \mathcal{L}(r_1(t),\ldots,r_M(t)|x,y)$$

$$\mathcal{L}(r_1(t), \ldots, r_M(t)|x, y) = \prod_{i=1}^{M} \mathcal{L}(r_i(t)|x, y)$$

$$r_i(t) = \sum_{j=1}^{K} A_{ij} s_j(t - \tau_{ij}(x, y)) + w_i(t)$$

$$= \boldsymbol{A}_i' \boldsymbol{S}(t - \boldsymbol{\tau}_i) + w_i(t)$$

$$\boldsymbol{A}_i = [A_{i1}, \ldots, A_{iK}]',$$

iid Gaussian $\mathcal{N}(0,\rho)$

$$\boldsymbol{S}(t - \boldsymbol{\tau}_i) = [s_1(t - \tau_{i1}), \ldots, s_K(t - \tau_{iK})]'$$

Known deterministic signals

Known deterministic functions of (x,y)

Assume we measure $r_i(t)$ during the interval $[0,T]$ then

$$\mathcal{L}(r_i(t)|x,y,\boldsymbol{A}_i) = e^{-0.5\boldsymbol{A}_i'\mathbf{Q}_i(x,y)\boldsymbol{A}_i + \boldsymbol{A}_i'\boldsymbol{R}_i(x,y)}$$

$$\mathbf{Q}_i(x,y) = \int_0^T \boldsymbol{S}(t-\boldsymbol{\tau}_i)\boldsymbol{S}'(t-\boldsymbol{\tau}_i)dt$$

$$\boldsymbol{R}_i(x,y) = \int_0^T r_i(t)\boldsymbol{S}(t-\boldsymbol{\tau}_i)dt$$

Integrating out $\boldsymbol{A}_i$

Plays the role of SNR

$$\mathcal{L}(r_i(t)|x,y) = \frac{e^{0.5\rho R_i'(x,y)\{\rho\mathbf{Q}_i(x,y)+\mathbf{I}\}^{-1}R_i(x,y)}}{\sqrt{\det(\rho\mathbf{Q}_i(x,y)+\mathbf{I})}}$$

$$\hat{\theta} = \hat{\theta}_{\mathrm{MSE}} = \frac{\int \theta \mathcal{L}(X|\theta)\pi(\theta)d\theta}{\int \mathcal{L}(X|\theta)\pi(\theta)d\theta}$$

Assuming **uniform** prior for the target position on the operational space of the radar and by sampling uniformly this space we have

$$\begin{bmatrix} \hat{x} \\ \hat{y} \end{bmatrix} \approx \frac{\sum_{(x_k,y_k)\in\mathrm{op.sp.}} \begin{bmatrix} x_k \\ y_k \end{bmatrix} \prod_{i=1}^{M} \mathcal{L}(r_i|x_k,y_k)}{\sum_{(x_k,y_k)\in\mathrm{op.sp.}} \prod_{i=1}^{M} \mathcal{L}(r_i|x_k,y_k)}$$

Simulations

$s_i(t)$ are sinusoids

Duration=10^{-4}sec

Integration period=$5x10^{-4}$sec

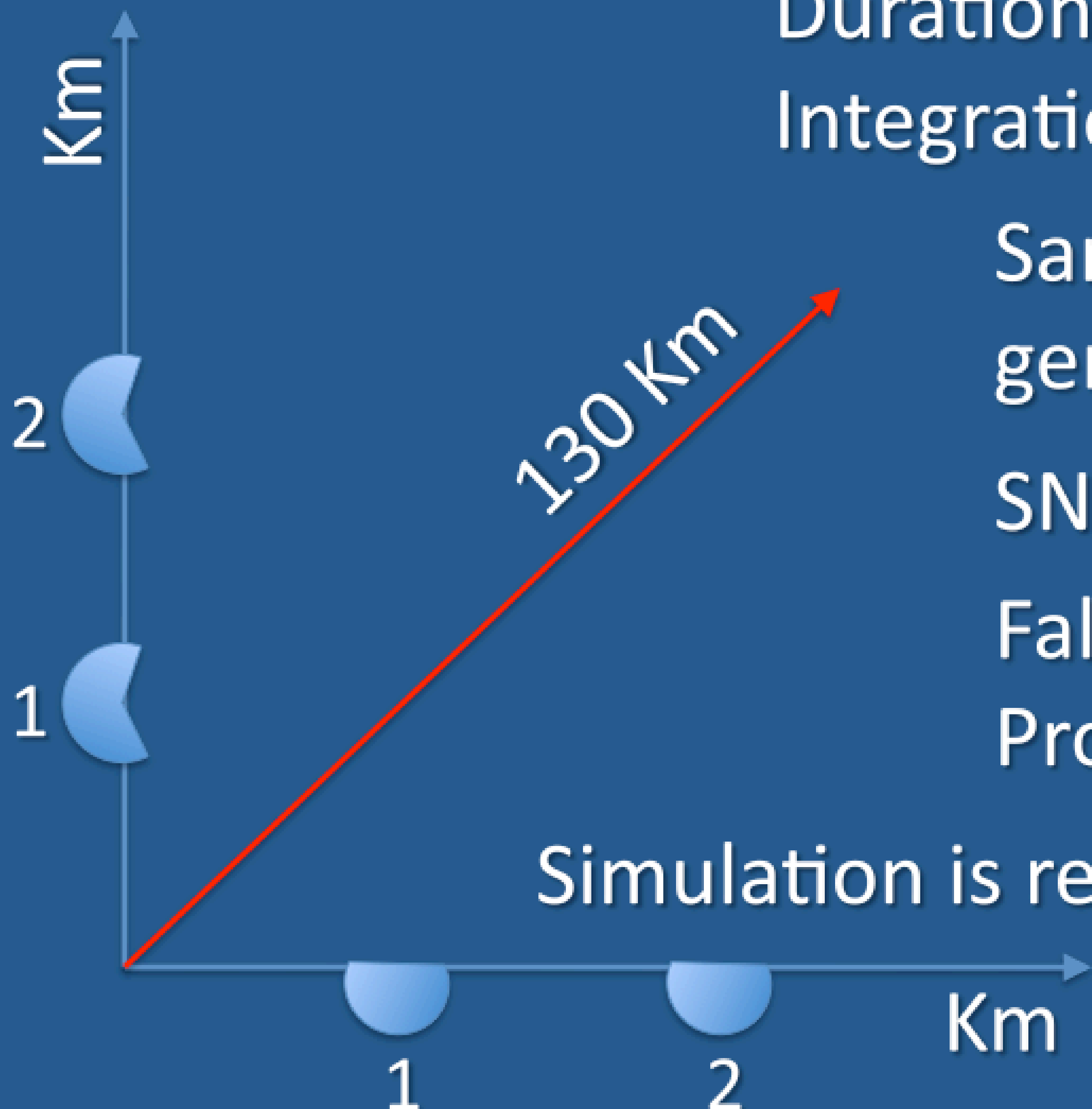
Sampling: dx=dy=10 Km
generates 146 points

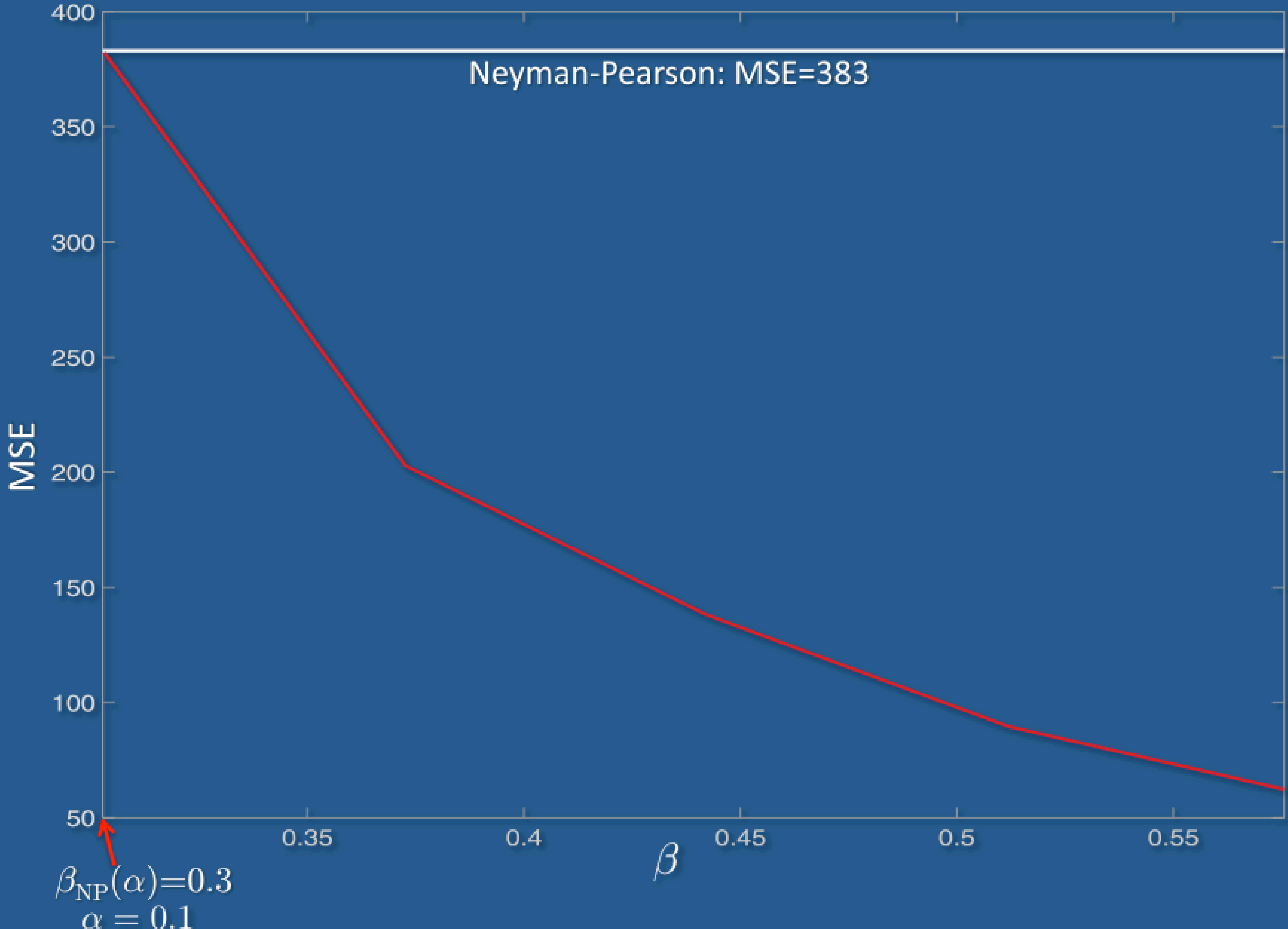
SNR=-10 dB

False alarm: α=0.1

Prob. Miss: $\beta_{\mathrm{NP}}(\alpha)$=0.3

Simulation is repeated 10,000 times




Neyman-Pearson: MSE=383
MSE
400
350
300
250
200
150
100
50
0.35
0.4
0.45
0.5
0.55
β
$\beta_{\mathrm{NP}}(\alpha)=0.3$
$\alpha = 0.1$