Sequential rate change detection in Poisson processes

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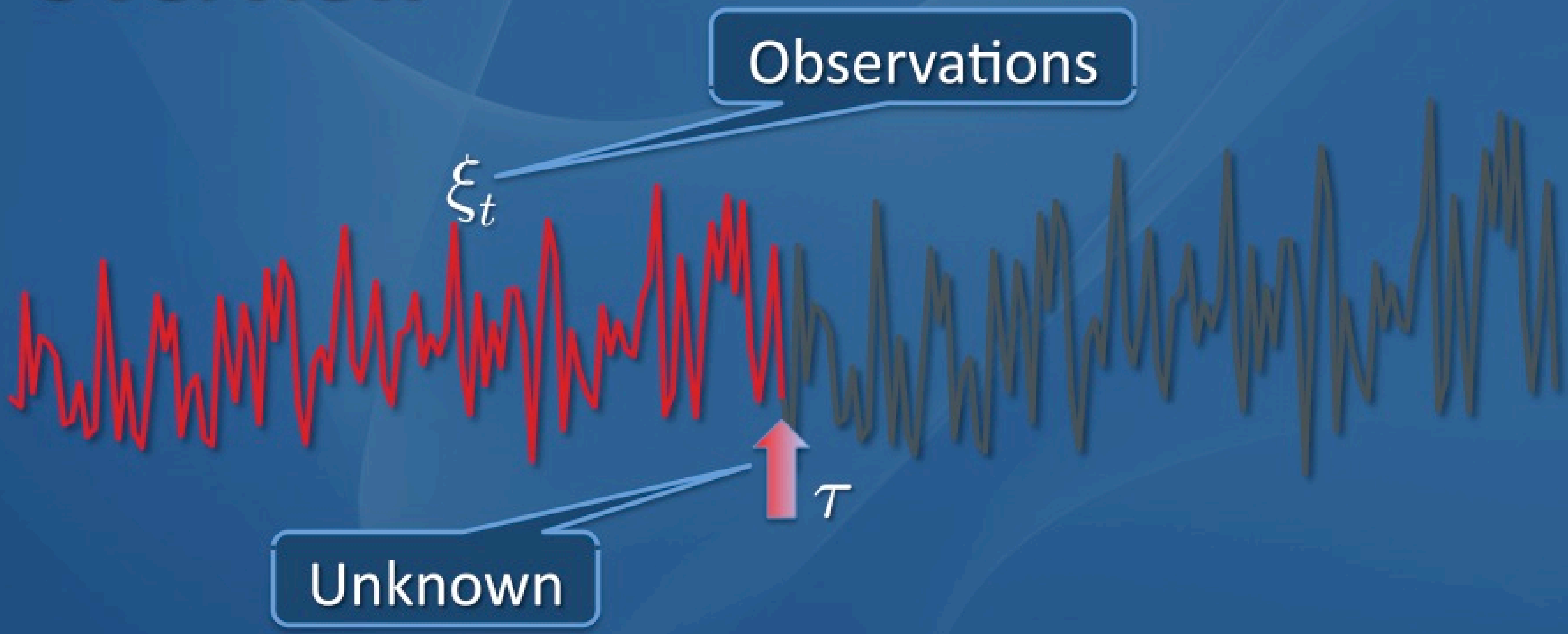
Outline

- Overview of sequential changepoint detection
- CUSUM test and Lorden's criterion
- The homogeneous Poisson disorder problem
  - CUSUM average run length
  - CUSUM optimality
- Non-homogeneous Poisson disorder problem and Epidemic surveillance

Overview



Using $\{\xi_t\}$ detect τ as soon as possible

Quality control

Systems monitoring

Remote sensing and GIS

Smart cameras – Human computer interaction

Image processing

⋮

Optical communications

Changepoint models for hazard functions

Occurrence of industrial accidents

Epidemic detection

Monitoring of link failures in computer networks

⋮

P_∞: nominal measure

P_0 : alternative measure

P_τ : measure induced by the change

E_τ [.]: corresponding expectation

Both P_0 and P_∞ are assumed known

We are interested in **sequential schemes**

With every new data point ξ_t we decide whether

- Stop and raise an alarm
- Continue sampling

For the decision at time t we use all the available information up to time t

$$\mathcal{F}_t = \sigma\{\,\xi_s : 0 \le s \le t\,\}$$

Sequential test $\Leftrightarrow$ stopping time T adapted to the filtration $\{\mathcal{F}_t\}$

Detection delay

what is τ ?

$$\mathcal{J}(T) = \mathsf{E}_\tau[T - \tau \,|\, T > \tau]$$

False alarm

Average period between false alarms:

$$\mathsf{E}_\infty[T] \geq \gamma$$

False alarm probability:

$$\mathsf{P}(T \leq \tau) \leq \alpha$$

$$\min_T \mathcal{J}(T)$$

Changepoint mechanism

There is a mechanism that **decides when** to impose the change. This decision can be

- **Independent** from the observations $\{\xi_t\}$
- **Depend** on the observations $\{\xi_t\}$

If independent, then τ appears as random variable with prior

$$\mathsf{P}(\tau = t) = \pi_t$$

If $\{\pi_t\}$ known then **Bayesian formulation**

Bayesian Formulation

Zero modified geometric

$$\pi_t = (1-\varpi)p(1-p)^{t-1}, t \geq 1; \quad \pi_0 = \varpi.$$

$$\min_T \mathsf{E}[T-\tau \,|\, T>\tau]; \text{ s.t. } \mathsf{P}(T \leq \tau) \leq \alpha$$

$$S_t = (S_{t-1}+1)\frac{f_0(\xi_t)}{(1-p)f_\infty(\xi_t)}, \quad S_0 = r$$

$$T_{\mathrm{SR}} = \inf\{t: \; S_t \geq \nu\}$$

I.i.d.: Shiryaev (1963,1978), Poor (1998). **BM**: Shiryaev (1961). **Poisson**: Peskir and Shiryaev (2002), Dayanik (2005,2006,2008)

The mechanism decides without consulting the observations but $\mathsf{P}(\tau = t) = \pi_t$ is **unknown**

$$\mathscr{J}_{\mathrm{P}}(T) = \sup_{\{\pi_t\}} \mathsf{E}_\tau[T - \tau | T > \tau]$$

$$= \sup_{t \geq 0} \mathsf{E}_t[T - t | T > t]$$

Pollak's (1985) criterion

$$\inf_T \mathscr{J}_{\mathrm{P}}(T) = \inf_T \sup_{t \geq 0} \mathsf{E}_t[T - t | T > t]$$

$$\text{s.t.} \quad \mathsf{E}_\infty[T] \geq \gamma$$

Pollak (1985) proposed the following stopping time known as Shiryaev-Roberts-Pollak

$$S_t = (S_{t-1} + 1)\frac{f_0(\xi_t)}{f_\infty(\xi_t)}, \quad S_0 \sim q(S)$$

$$T_{\mathrm{SRP}} = \inf\{t : S_t \geq \nu\}$$

It is an equalizer

$$\mathsf{E}_t[T_{\mathrm{SRP}} - t | T_{\mathrm{SRP}} > t] = \text{constant}$$

$$\mathcal{J}_{\mathrm{P}}(T_{\mathrm{SRP}}) - \inf_T \mathcal{J}_{\mathrm{P}}(T) = o(1), \quad \text{as } \gamma \to \infty$$

Strict Optimality ?

In 1997 appears a proof that the SRP test is optimum (Annals of Statistics)

Yajun Mei (2006), shows that the proof is problematic

The conjecture remained unanswered until last year (2010): Tartakovsky and Polunchenko, produced a counterexample. After 25 years we can finally say that

The SRP test is NOT optimum

CUSUM and Lorden's criterion

The changepoint mechanism decides to impose the change by consulting the observations $\{\xi_t\}$ and possibly additional information.

In this case τ becomes a stopping time adapted to a larger filtration than $\{\mathcal{F}_t\}$.

Lorden's (1971) criterion

$$\mathcal{J}_{\mathrm{L}}(T) = \sup_{\tau} \mathsf{E}_{\tau}[T - \tau | T > \tau]$$

$$= \sup_{t \geq 0} \operatorname{essup} \mathsf{E}_t[T - t | T > t, \mathcal{F}_t]$$

$$X_t = AX_{t-1} + BW_t$$

State space model

$$\xi_t = CX_t + Dw_t$$

Measurements

Possible cause for change is for example "large" oscillations

$$\tau = \inf\{t : \ \|X_t\| \geq c\}$$

$$A \rightarrow A'$$

A bomb set to explode at a specific time

$$\inf_T \mathcal{J}_{\rm L}(T) = \inf_T \sup_{t\geq 0} \operatorname{essup} \mathsf{E}_t[T-t|T>t,\mathcal{F}_t]$$

$$\mathsf{E}_\infty[T] \geq \gamma$$

The optimum scheme: CUSUM

$$u_t = \log\left(\frac{d\mathsf{P}_0}{d\mathsf{P}_\infty}(\mathcal{F}_t)\right)$$

$$m_t = \inf_{0\leq s\leq t} u_s$$

$$y_t = u_t - m_t \geq 0$$

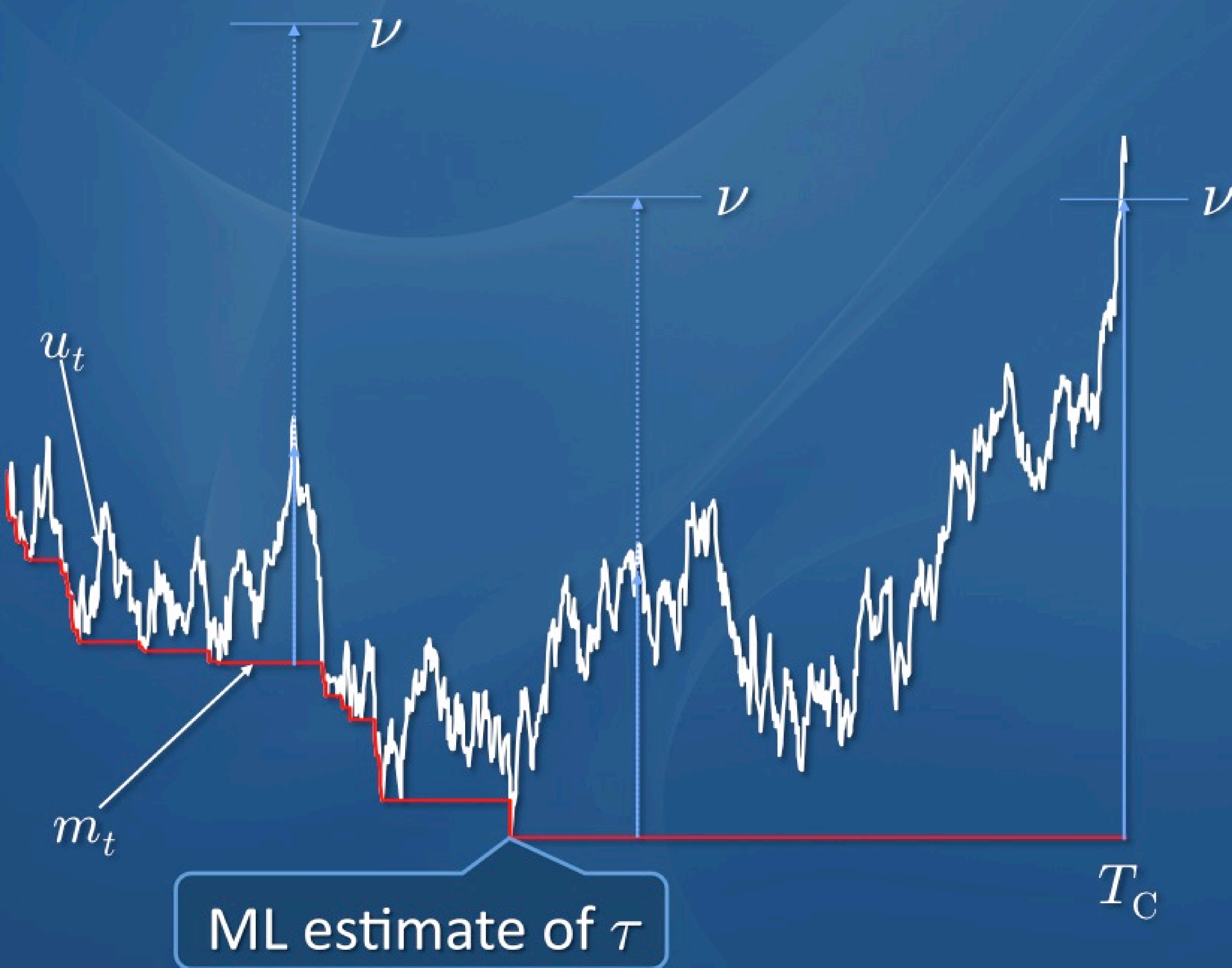
$$T_{\rm C} = \inf\{t: \; y_t \geq \nu\}$$

I.i.d.: Moustakides (1986)

Ritov (1990), Poor (1998)

BM: Shiryaev (1996), Beibel (1996)

Ito: Moustakides (2004)



ν
ν
ν
u_t
m_t
ML estimate of τ
T_C

Homogeneous Poisson disorder

Let $\{\mathcal{N}_t\}$ denote a homogeneous Poisson process with rate λ satisfying

$$\lambda = \begin{cases} \lambda_\infty & \text{for } t \leq \tau \\ \lambda_0 & \text{for } t > \tau. \end{cases}$$

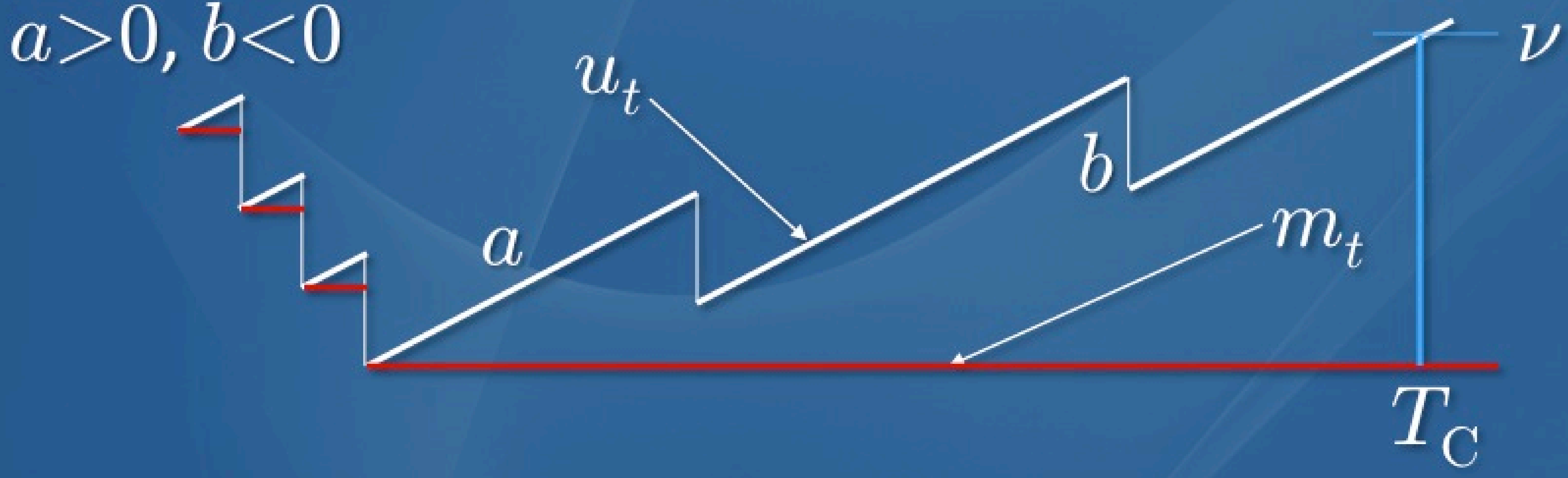
$$u_t = (\lambda_\infty - \lambda_0)t + \log \frac{\lambda_0}{\lambda_\infty} \mathcal{N}_t$$

$$u_t = a\,t + b\,\mathcal{N}_t$$

a,b opposite signs

Show optimality of CUSUM

First step compute ARL: $\mathsf{E}_0[T_C]$ and $\mathsf{E}_\infty[T_C]$



We are looking for $f(y)=\mathsf{E}[T_C \,|\, y_0=y]$

$$\mathsf{E}[\underbrace{f(y_{T_C})}_{=0}] - f(y_0) =$$

$$\mathsf{E}\left[\int_0^{T_C} \underbrace{\{af'(y_{t-}) + \lambda[f(y_{t-}+b) - f(y_{t-})]\}}_{=-1} dt\right]$$

$$af'(y) + \lambda[f(y+b) - f(y)] = -1$$

$$f(\nu) = 0;\ f(y) = f(0)\ y \leq 0$$

$$f(y) = \frac{1}{\lambda}\left\{\sum_{n=0}^{[\frac{\nu}{|b|}]} \phi_n(\nu - n|b|) - \sum_{n=0}^{[\frac{y}{|b|}]} \phi_n(y - n|b|)\right\}$$

$$\phi_n(y) = e^{\frac{\lambda y}{a}}\left(\sum_{k=0}^{n} \frac{\left(\frac{-\lambda y}{a}\right)^k}{k!}\right) - 1;$$

Optimality of CUSUM

$$\mathcal{J}_{\mathrm{L}}(T) = \sup_{t\geq 0} \operatorname{essup} \mathsf{E}_t[T - t | T > t, \mathcal{F}_t]$$

$$\mathsf{E}_\infty[T] \geq \gamma$$

$$\mathsf{E}_\infty[T] \geq \mathsf{E}_\infty[T_{\mathrm{C}}] = \gamma \quad \Rightarrow \mathcal{J}_{\mathrm{L}}(T) \geq \mathcal{J}_{\mathrm{L}}(T_{\mathrm{C}})$$

$$h(y) = \mathsf{E}_\infty[T_{\mathrm{C}} | y_0 = y]$$

$$g(y) = \mathsf{E}_0[T_{\mathrm{C}} | y_0 = y]$$

Equalizer

$$\operatorname{essup} \mathsf{E}_t[T_{\mathrm{C}} - t | T_{\mathrm{C}} > t, \mathcal{F}_t] = g(0)$$

$$\mathcal{J}_{\mathrm{L}}(T_{\mathrm{C}}) = g(0) \qquad \mathsf{E}_\infty[T_{\mathrm{C}}] = h(0)$$

$$\mathsf{E}_\infty[T] \ge h(0) \Rightarrow \mathcal{J}_{\mathrm{L}}(T) \ge g(0)$$

Lemma: $$\mathcal{J}_{\mathrm{L}}(T) \ge \frac{\mathsf{E}_\infty[\int_0^T e^{y_t}\,dt]}{\mathsf{E}_\infty[e^{y_T}]}$$

$$\mathsf{E}_\infty[T] \ge h(0) \Rightarrow \frac{\mathsf{E}_\infty[\int_0^T e^{y_t}\,dt]}{\mathsf{E}_\infty[e^{y_T}]} \ge g(0)$$

$$\mathsf{E}_\infty[\int_0^T e^{y_t}\,dt] = g(0)\mathsf{E}_\infty[e^{y_T}] + \underbrace{\mathsf{E}_\infty[T] - h(0)}_{\ge 0}$$

$$+\underbrace{\mathsf{E}_\infty[h(y_T) - e^{y_T} g(y_T)]}_{\ge 0}$$

Lemma: $$h(y) - e^{y} g(y) \ge 0, \quad \forall\ y \ge 0$$

Nonhomogeneous Poisson

Let $\{\mathcal{N}_t\}$ denote a nonhomogeneous Poisson process with rate λ_t that satisfies

$$\lambda_t = \begin{cases} \omega_t & \text{for } t \leq \tau \\ \rho\,\omega_t & \text{for } t > \tau, \end{cases}$$

ω_t is adapted to our observations (that can include more information than $\{\mathcal{N}_t\}$) and ρ is a known constant.

CUSUM test:

$$u_t = (1-\rho)\int_0^t \omega_s\, ds + (\log\rho)\mathcal{N}_t$$

$$m_t = \inf_{0\le s\le t} u_s$$

$$y_t = u_t - m_t$$

$$T_{\rm C} = \inf\{t:\ y_t \ge \nu\}$$

Is it optimum? **YES !** but in a Lorden-*like* sense

$$\mathcal{J}_{\rm L}(T) = \sup_{t\ge 0} \operatorname{essup} \mathsf{E}_t[\mathcal{N}_T - \mathcal{N}_t | T > t, \mathcal{F}_t]$$

$$\mathsf{E}_\infty[\mathcal{N}_T] \ge \gamma$$

Epidemic Detection

Detect the onset of an epidemic: when the **incidence rate** of a particular disease increases significantly above some standard level

ω_t : describes the nominal incidence rate, which can be a function of several observable quantities as population, pollution measurements etc.

$\rho > 1$: smallest increase in incidence rate that should be considered … alarming.

Detect rapidly **in number of incidences**