Optimum Joint Detection & Estimation

Application to MIMO Radar

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Outline

- Joint Detection and Estimation: problem formulation
- Optimum one- and two-step strategies
- MIMO radar
- Application of the two-step scheme
- Simulations

Joint Detection and Estimation

The problem: For a finite sequence of samples $X{=}[x_1,\ldots,x_n]$ we assume the following two hypotheses:

$$\mathsf{H}_0 : \; X \sim f_0(X)$$

$$\mathsf{H}_1 : \; X \sim f_1(X|\theta)$$

Detection: Given the data vector X, **decide** between the two hypotheses

Estimation: Every time there is a decision in favor of H_1 **estimate** θ

Radar

If a target enters the operational space of our radar, we would like to **detect** it.

Once a target is detected we would also like to **estimate** its position, speed…

Detection problem (Neyman-Pearson)

We are given a data vector $X=[x_1,\ldots,x_n]$ for which we assume the following two hypotheses:

$$\mathsf{H}_0 : \; X \sim f_0(X)$$
$$\mathsf{H}_1 : \; X \sim f_1(X)$$

Use X to select D $=$ H_0 or H_1

False alarm $\mathsf{P}_0(\mathsf{D}=\mathsf{H}_1)\leqslant \alpha$, $0<\alpha<1$

Maximize $\mathsf{P}_1(\mathsf{D}=\mathsf{H}_1)$

Minimize $\mathsf{P}_1(\mathsf{D}=\mathsf{H}_0)$

$$\frac{f_1(X)}{f_0(X)} \underset{\mathsf{H}_0}{\overset{\mathsf{H}_1}{\gtreqless}} t$$

$$\mathsf{H}_0 : \; X \sim f_0(X)$$
$$\mathsf{H}_1 : \; X \sim f_1(X|\theta), \quad \pi(\theta)$$

False alarm $\mathsf{P}_0(\mathsf{D} = \mathsf{H}_1) \leqslant \alpha$, $0 < \alpha < 1$

Minimize $\mathsf{P}_1(\mathsf{D} = \mathsf{H}_0)$

$$\frac{\int f_1(X|\theta)\pi(\theta)d\theta}{f_0(X)} \underset{\mathsf{H}_0}{\overset{\mathsf{H}_1}{\gtreqless}} t$$

$$\mathsf{P}_1(\mathsf{D} = \mathsf{H}_0) \geqslant \min \mathsf{P}_1(\mathsf{D} = \mathsf{H}_0) = \beta_{\mathrm{NP}}(\alpha)$$

Estimation problem (Bayesian approach)

We are given a data vector $X=[x_1,\ldots,x_n]$ for which we assume the following model:

$$X \sim f_1(X|\theta), \quad \pi(\theta)$$

Use X to find an estimate $\hat{\theta}$ of θ

Minimize $\mathsf{E}_1\left[\|\hat{\theta}-\theta\|^2\right]$

$$\hat{\theta}_o(X) = \mathsf{E}_1[\theta|X] = \frac{\int \theta f_1(X|\theta)\pi(\theta)d\theta}{\int f_1(X|\theta)\pi(\theta)d\theta}$$

Formulation of the joint problem

$$\mathsf{H}_0: \; X \sim f_0(X)$$

$$\mathsf{H}_1: \; X \sim f_1(X|\theta), \quad \pi(\theta)$$

Ad-hoc

Optimum in each subproblem:

$$\frac{\int f_1(X|\theta)\pi(\theta)d\theta}{f_0(X)} \underset{\mathsf{H}_0}{\overset{\mathsf{H}_1}{\gtreqless}} t$$

$$\hat{\theta}_o = \frac{\int \theta f_1(X|\theta)\pi(\theta)d\theta}{\int f_1(X|\theta)\pi(\theta)d\theta}$$

GLRT: $$\max_\theta \frac{f_1(X|\theta)}{f_0(X)} \underset{\mathsf{H}_0}{\overset{\mathsf{H}_1}{\gtreqless}} t$$

One-step tests

We distinguish our decisions into H_0 and H_1 **BUT**

- H_1 : detection and reliable estimation
- H_0 : no detection or detection without reliable estimation

$$\text{Minimize} \quad E_1\left[\|\hat{\theta}-\theta\|^2 | D = H_1\right]$$

We need to **control** the detection part

$$\alpha \geqslant P_0(D{=}H_1) \qquad \beta \geqslant P_1(D = H_0) \geqslant \beta_{NP}(\alpha)$$

Theorem: The optimum combined detection and estimation scheme is defined as follows:

$$\hat{\theta}_o(X) = \frac{\int \theta f_1(X|\theta)\pi(\theta)d\theta}{\int f_1(X|\theta)\pi(\theta)d\theta}$$

$$\sigma^2(X) = \mathsf{E}_1\left[\|\theta - \hat{\theta}_o\|^2 | X\right]$$

$$= \frac{\int \|\theta - \hat{\theta}_o\|^2 f_1(X|\theta)\pi(\theta)d\theta}{\int f_1(X|\theta)\pi(\theta)d\theta}$$

$$\frac{\int f_1(X|\theta)\pi(\theta)d\theta}{f_0(X)} \left\{\lambda - \sigma^2(X)\right\} \underset{\mathsf{H}_0}{\overset{\mathsf{H}_1}{\gtreqless}} t$$

Segmentation problems

Detect if after some point, there is a change in the statistical behavior of the data.

Every time we detect a change, we like to **estimate** the point of change, i.e. the **boundary**.

Detect objects in images and find boundaries. Detection has no meaning without boundary estimation.

Two-step tests

We distinguish our decisions into H_0 and H_1

Every time I decide in favor of H_1 I compute an estimate for my parameters

I can now ask myself: **Can I trust my estimate?**

Is the estimate reliable (denoted as H_{1r}) or unreliable (denoted as H_{1u}) ?

Need a **second decision mechanism** that decides between H_{1r} and H_{1u}

We have three decisions: H_0, H_{1r} and H_{1u} (the latter two grouped as H_1)

Decide H_0 / H_1

$$\alpha \geqslant \mathsf{P}_0(\mathsf{D}{=}\mathsf{H}_1) \quad \min \mathsf{P}_1(\mathsf{D}{=}\mathsf{H}_0)$$

$$\frac{\int f_1(X|\theta)\pi(\theta)d\theta}{f_0(X)} \underset{\mathsf{H}_0}{\overset{\mathsf{H}_1}{\gtreqless}} t$$

IF H_1

$$\min \mathsf{E}_1[\|\hat{\theta}-\theta\|^2|\mathsf{D}=\mathsf{H}_{1r}]$$

$$\frac{1-\beta}{1-\beta_{\mathrm{NP}}(\alpha)} \leq \frac{\mathsf{P}(\mathsf{D}=\mathsf{H}_{1r})}{\mathsf{P}(\mathsf{D}=\mathsf{H}_1)}$$

Estimate θ

$$\hat{\theta}_o(X) = \frac{\int \theta f_1(X|\theta)\pi(\theta)d\theta}{\int f_1(X|\theta)\pi(\theta)d\theta}$$

$$\sigma^2(X) \underset{\mathsf{H}_{1u}}{\overset{\mathsf{H}_{1r}}{\lesseqgtr}} \lambda$$

Decide H_{1r} / H_{1u}

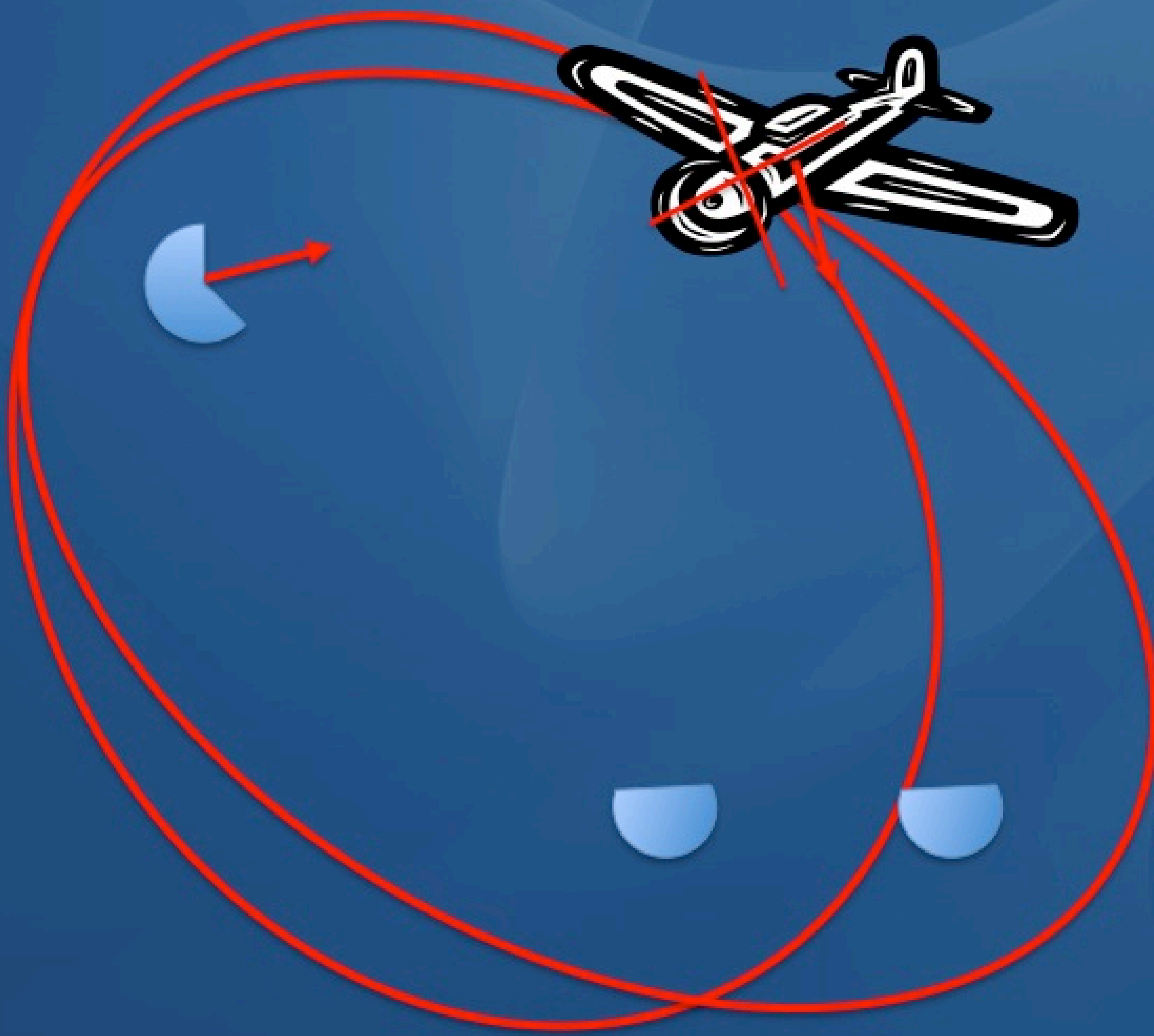
$$\mathcal{L}(X|\theta) = \frac{f_1(X|\theta)}{f_0(X)}$$

$$\frac{\int f_1(X|\theta)\pi(\theta)d\theta}{f_0(X)} = \int \mathcal{L}(X|\theta)\pi(\theta)d\theta \underset{\mathsf{H}_0}{\overset{\mathsf{H}_1}{\gtreqless}} t$$

$$\hat{\theta}_o(X) = \frac{\int \theta\mathcal{L}(X|\theta)\pi(\theta)d\theta}{\int \mathcal{L}(X|\theta)\pi(\theta)d\theta}$$

$$\sigma^2(X) = \frac{\int \|\theta - \hat{\theta}_o\|^2\mathcal{L}(X|\theta)\pi(\theta)d\theta}{\int \mathcal{L}(X|\theta)\pi(\theta)d\theta}$$

MIMO radar

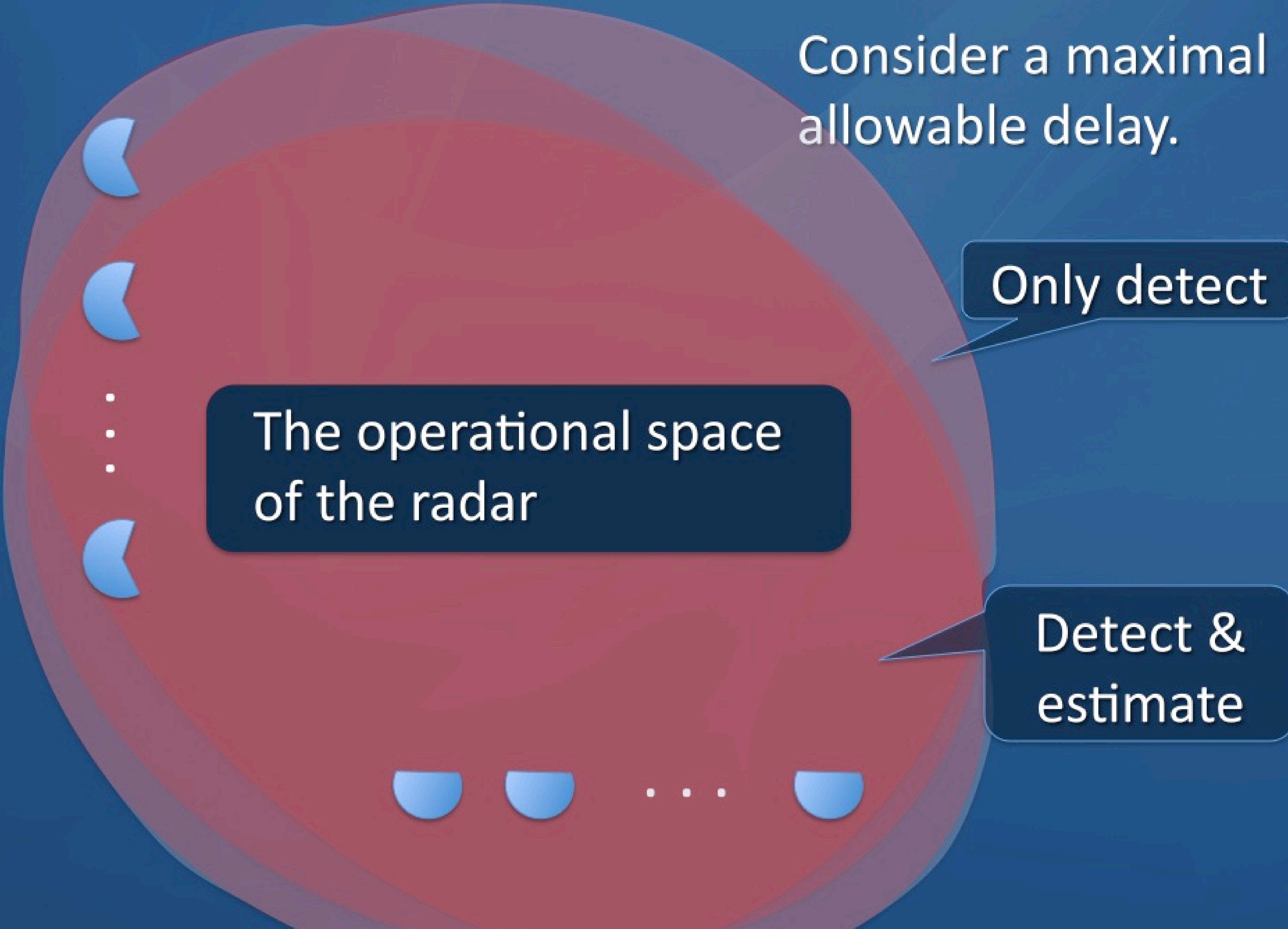


Measure the delay.

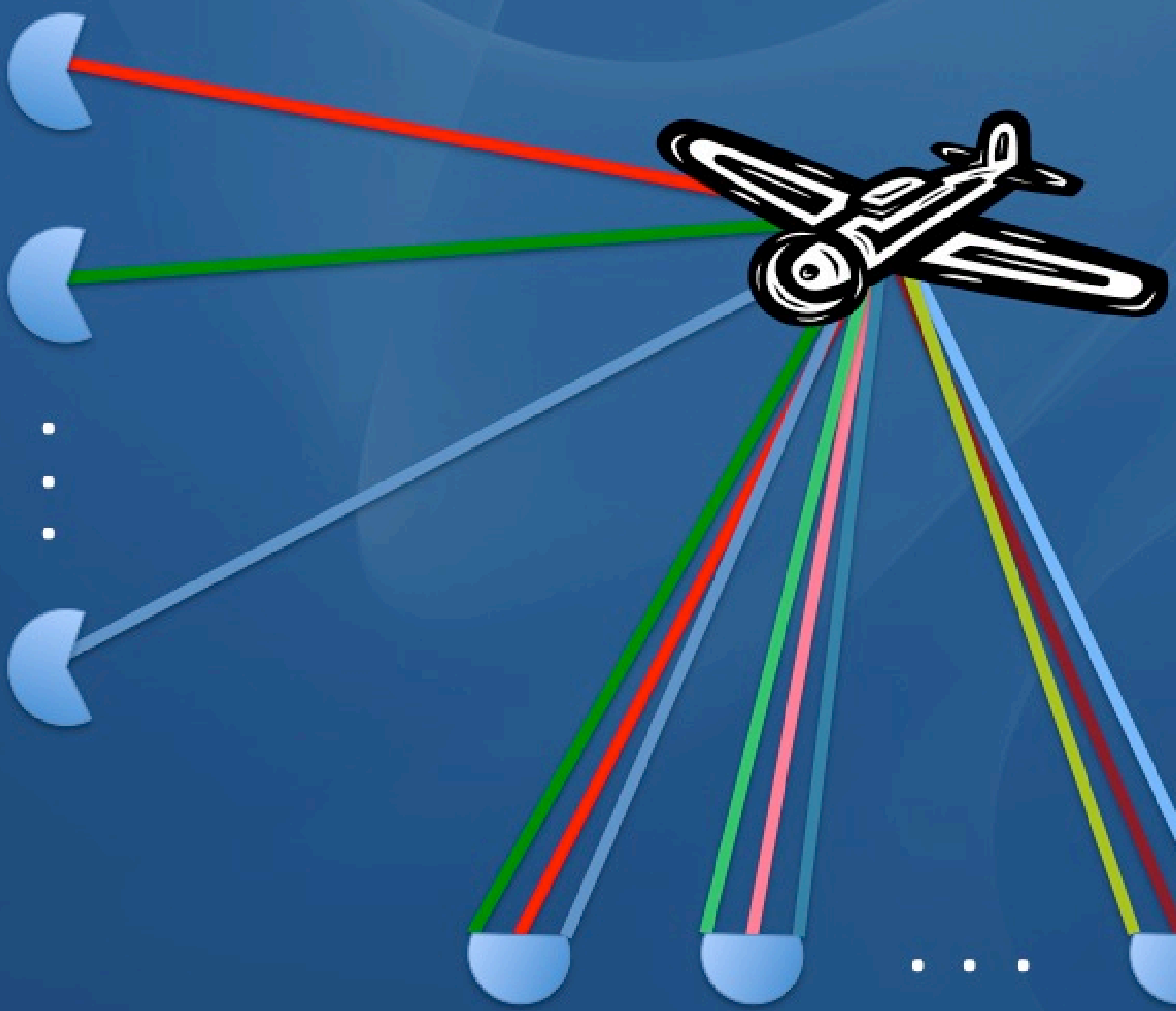
The delay is proportional to the distance traveled by the EM wave.

Defines an ellipse.

Consider a maximal allowable delay.

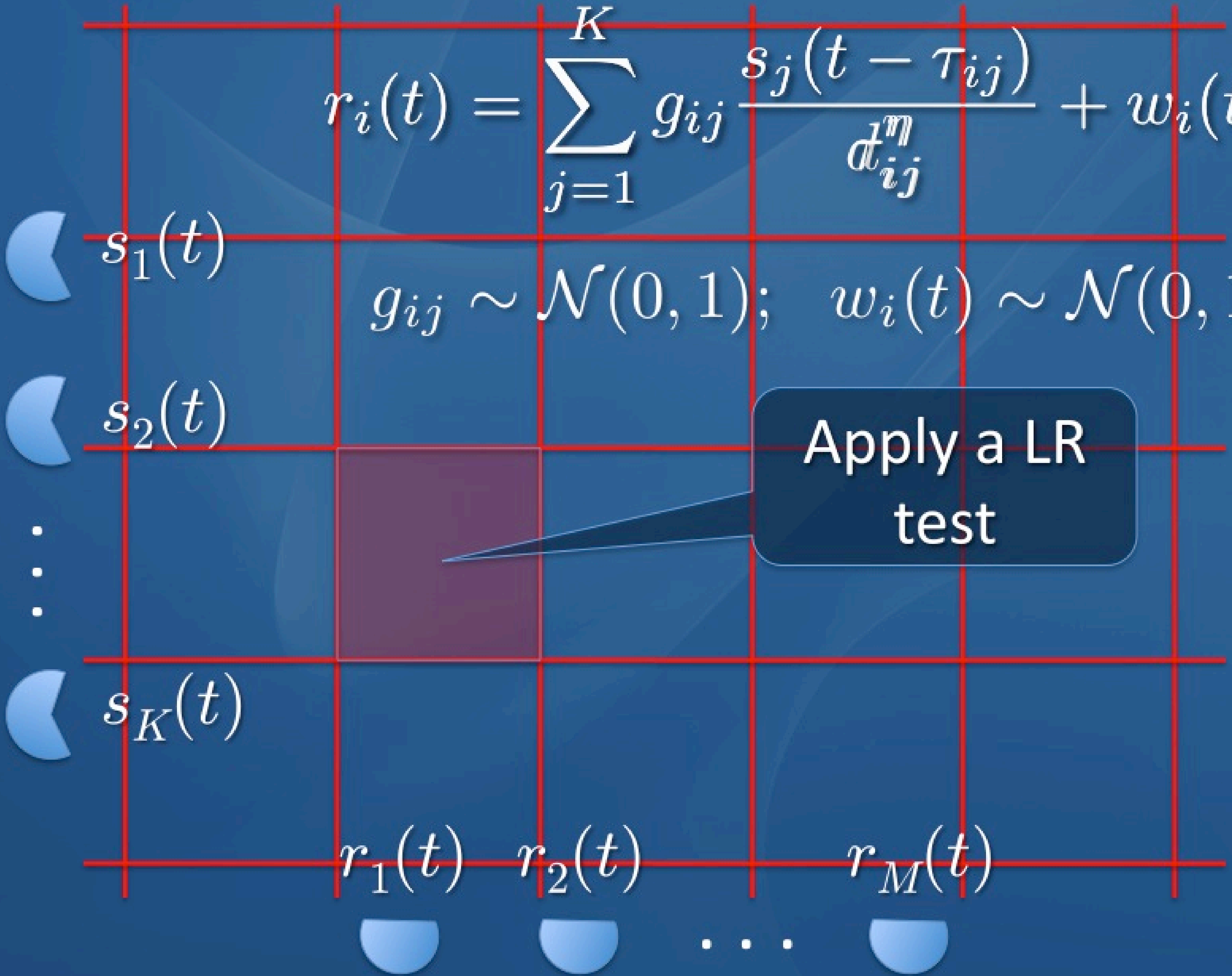


Application of optimum scheme



$$r_i(t) = \sum_{j=1}^{K} g_{ij} \frac{s_j(t - \tau_{ij})}{d_{ij}^{m}} + w_i(t)$$

$$g_{ij} \sim \mathcal{N}(0,1); \quad w_i(t) \sim \mathcal{N}(0,1)$$



$$r_i(t) = \sum_{j=1}^{K} g_{ij} \frac{s_j(t-\tau_{ij})}{\tau_{ij}^{\eta}} + w_i(t)$$

$$g_{ij} \sim \mathcal{N}(0,1); \quad w_i(t) \sim \mathcal{N}(0,1)$$

$s_1(t)$

$s_2(t)$

⋮

$s_K(t)$

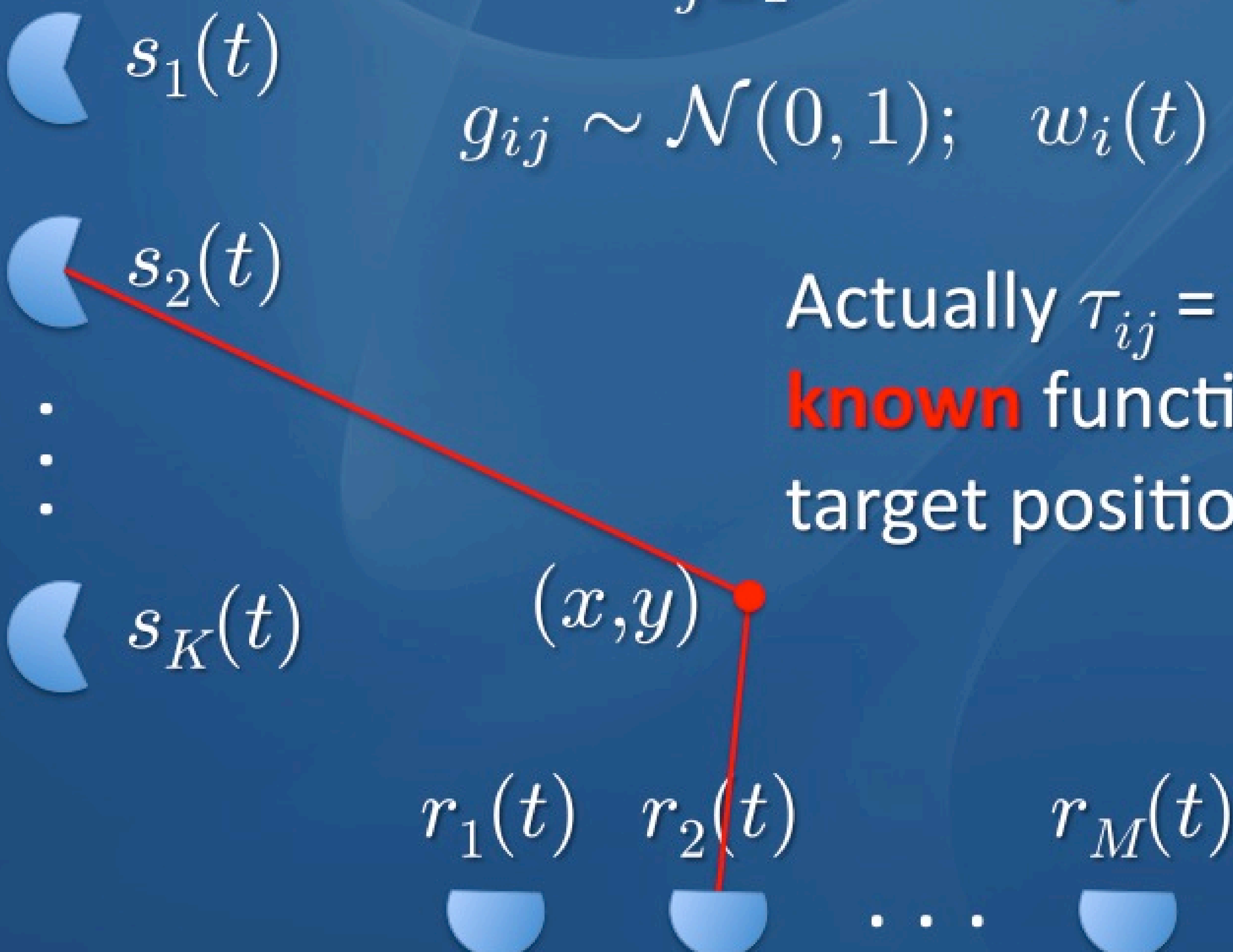
Treat the delays τ_{ij} as unrelated and estimate them

The KN delays generate KN ellipses and a set of at most KN choose 2 target positions. We average.

$r_1(t)$ $r_2(t)$ . . . $r_M(t)$

$$r_i(t) = \sum_{j=1}^{K} g_{ij} \frac{s_j(t-\tau_{ij})}{\tau_{ij}^{\eta}} + w_i(t)$$

$$g_{ij} \sim \mathcal{N}(0,1); \quad w_i(t) \sim \mathcal{N}(0,1)$$



Actually $\tau_{ij} = \tau_{ij}(x,y)$ are **known** functions of the target position

$$\mathsf{H}_0 : r_i(t) = w_i(t)$$

$$\mathsf{H}_1 : r_i(t) = \sum_{j=1}^{K} g_{ij} \frac{s_j(t-\tau_{ij})}{\tau_{ij}^{\eta}} + w_i(t)$$

$$g_{ij} \sim \mathcal{N}(0,1); \quad w_i(t) \sim \mathcal{N}(0,1)$$

For the position (x,y) we assume **uniform** prior over the operational space of the radar.

$$X \Longleftrightarrow [r_1(t),\ldots,r_M(t)] \qquad \theta \Longleftrightarrow (x,y)$$

$$\mathcal{L}(X|\theta) \Longleftrightarrow \mathcal{L}(r_1(t),\ldots,r_M(t)|x,y)$$

$$\mathcal{L}(r_1(t),\ldots,r_M(t)|x,y) = \prod_{i=1}^{M} \mathcal{L}(r_i(t)|x,y)$$

$$r_i(t) = \sum_{j=1}^{K} g_{ij} \frac{s_j(t-\tau_{ij})}{\tau_{ij}^{\eta}} + w_i(t)$$

$$= G_i' \, S(t-\mathcal{T}_i) + w_i(t)$$

$$G_i = [g_{i1},\ldots,g_{iK}]',$$

iid Gaussian $\mathcal{N}(0,1)$

$$S(t-\mathcal{T}_i) = \left[\frac{s_1(t-\tau_{i1})}{\tau_{i1}^{\eta}},\ldots,\frac{s_K(t-\tau_{iK})}{\tau_{iK}^{\eta}}\right]'$$

Known deterministic signals

Known deterministic functions of (x,y)

Assume we measure $r_i(t)$ during the interval $[0,T]$ then

$$\mathcal{L}(r_i(t)|x,y,G_i) = e^{-0.5G_i'\mathbf{Q}_i(x,y)G_i + G_i'R_i(x,y)}$$

$$\mathbf{Q}_i(x,y) = \int_0^T S(t-\mathcal{T}_i)S'(t-\mathcal{T}_i)dt$$

$$R_i(x,y) = \int_0^T r_i^*(t)S(t-\mathcal{T}_i)dt$$

Integrating out G_i

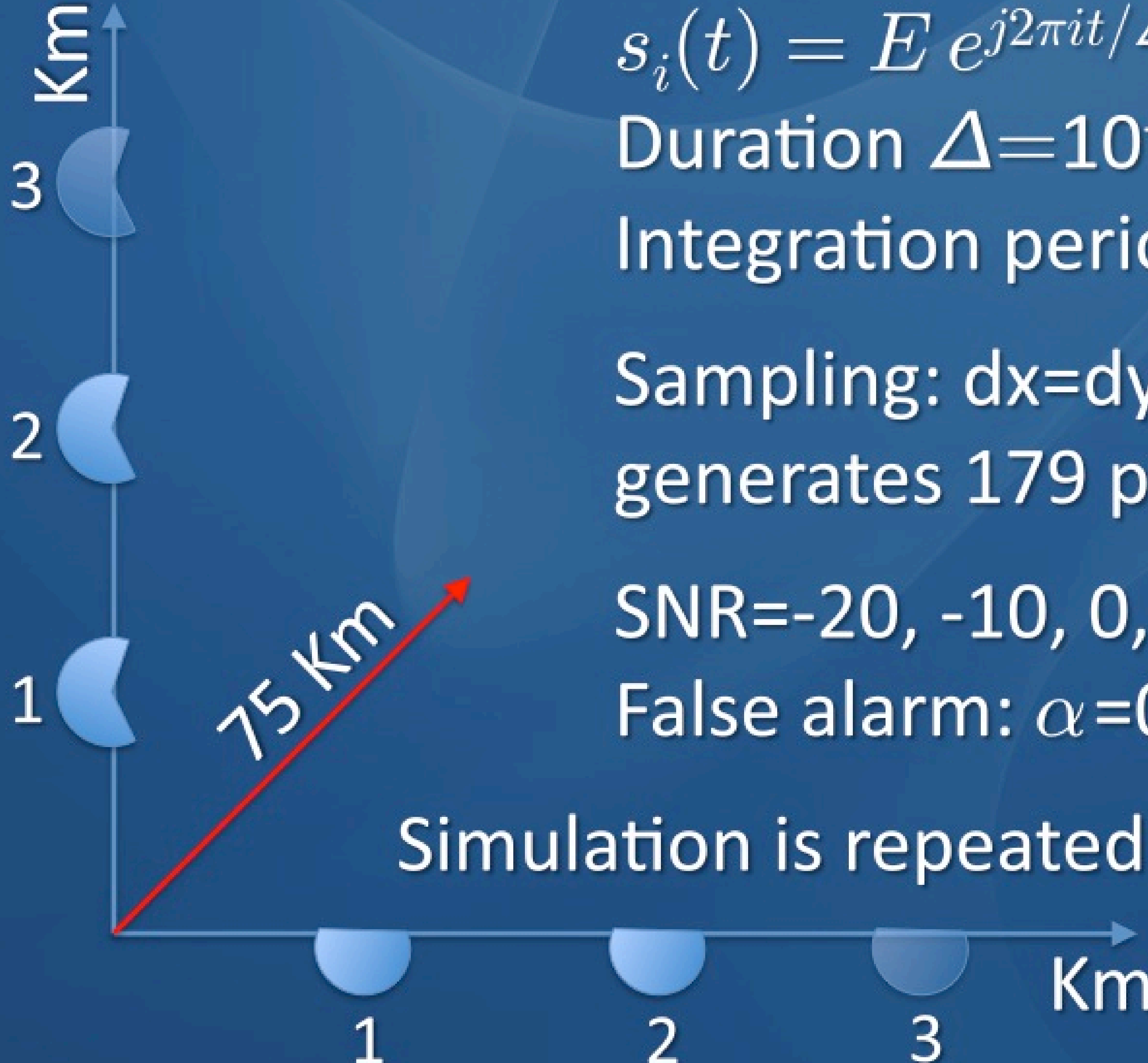
$$\mathcal{L}(r_i(t)|x,y) = \frac{e^{0.5R_i'(x,y)\{\mathbf{Q}_i(x,y)+\mathbf{I}\}^{-1}R_i(x,y)}}{\det(\mathbf{Q}_i(x,y)+\mathbf{I})}$$

$$\hat{\theta}_o = \frac{\int \theta \mathcal{L}(X|\theta)\pi(\theta)d\theta}{\int \mathcal{L}(X|\theta)\pi(\theta)d\theta}$$

Assuming **uniform** prior for the target position on the operational space of the radar and by sampling uniformly this space we have

$$\begin{bmatrix} \hat{x} \\ \hat{y} \end{bmatrix} \approx \frac{\displaystyle\sum_{(x_k,y_k)\in \text{op.sp.}} \begin{bmatrix} x_k \\ y_k \end{bmatrix} \prod_{i=1}^{M} \mathcal{L}(r_i|x_k,y_k)}{\displaystyle\sum_{(x_k,y_k)\in \text{op.sp.}} \prod_{i=1}^{M} \mathcal{L}(r_i|x_k,y_k)}$$

Simulations



$$s_i(t) = E\, e^{j2\pi it/\Delta}$$

Duration $\Delta = 10^{-4}$sec

Integration period $T = 5\text{x}10^{-4}$sec

Sampling: dx=dy=10 Km

generates 179 points

SNR=-20, -10, 0, 10 dB

False alarm: α=0.001

Simulation is repeated 200,000 times

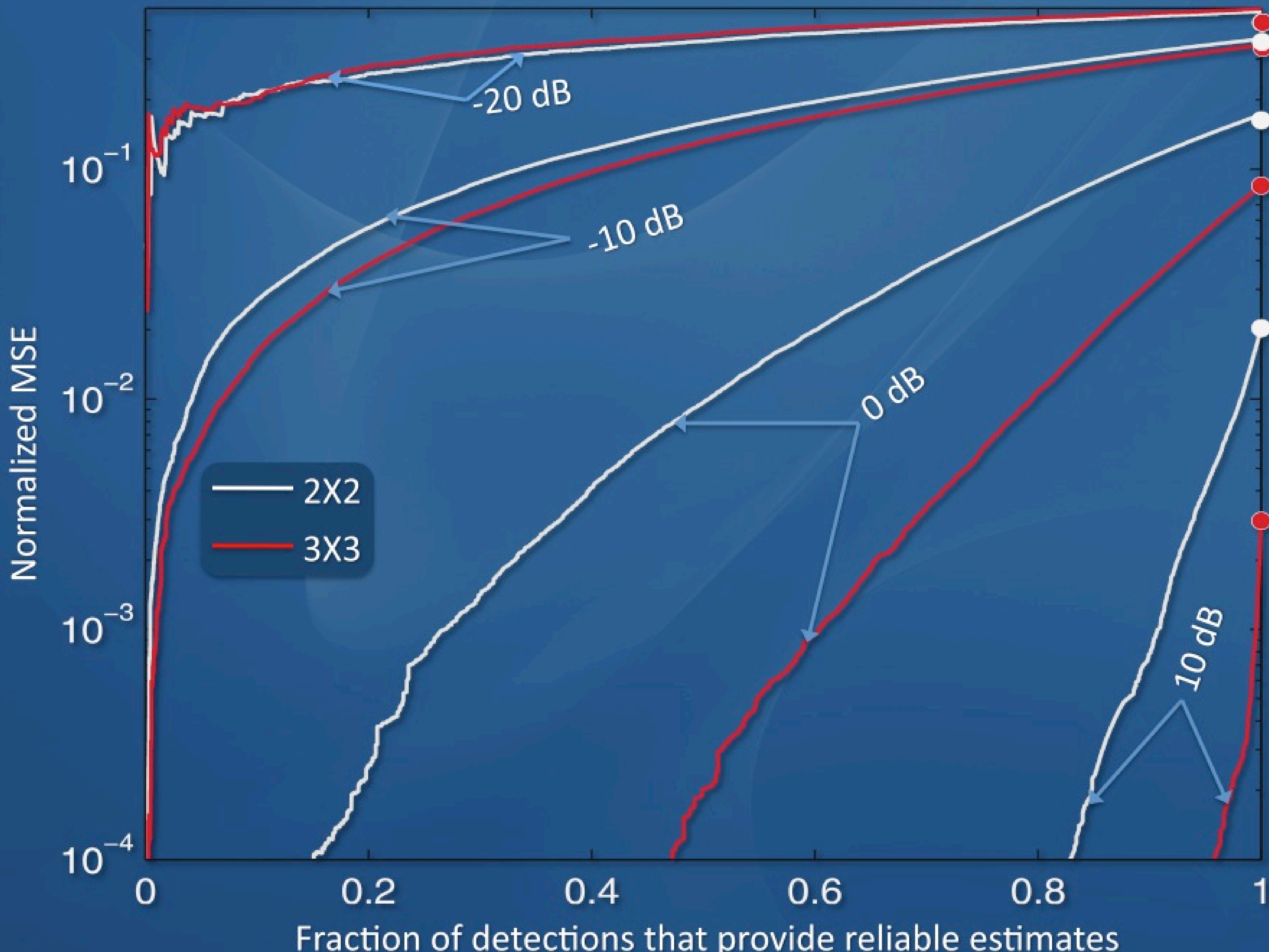
1) GLRT: $$\max_{x,y} \prod_{i=1}^{M} \mathcal{L}(r_i|x,y) \underset{\mathsf{H}_0}{\overset{\mathsf{H}_1}{\gtreqless}} t$$

$$\left[\begin{array}{c} \hat{x} \\ \hat{y} \end{array}\right] = \arg\max_{x,y} \prod_{i=1}^{M} \mathcal{L}(r_i|x,y)$$

2) For each subproblem use the optimum.

3) Proposed optimum two-step test.

We note that 2) is a special case of 3)


-20 dB
-10 dB
0 dB
10 dB
2X2
3X3
Normalized MSE
Fraction of detections that provide reliable estimates
0
0.2
0.4
0.6
0.8
1
10^{-1}
10^{-2}
10^{-3}
10^{-4}