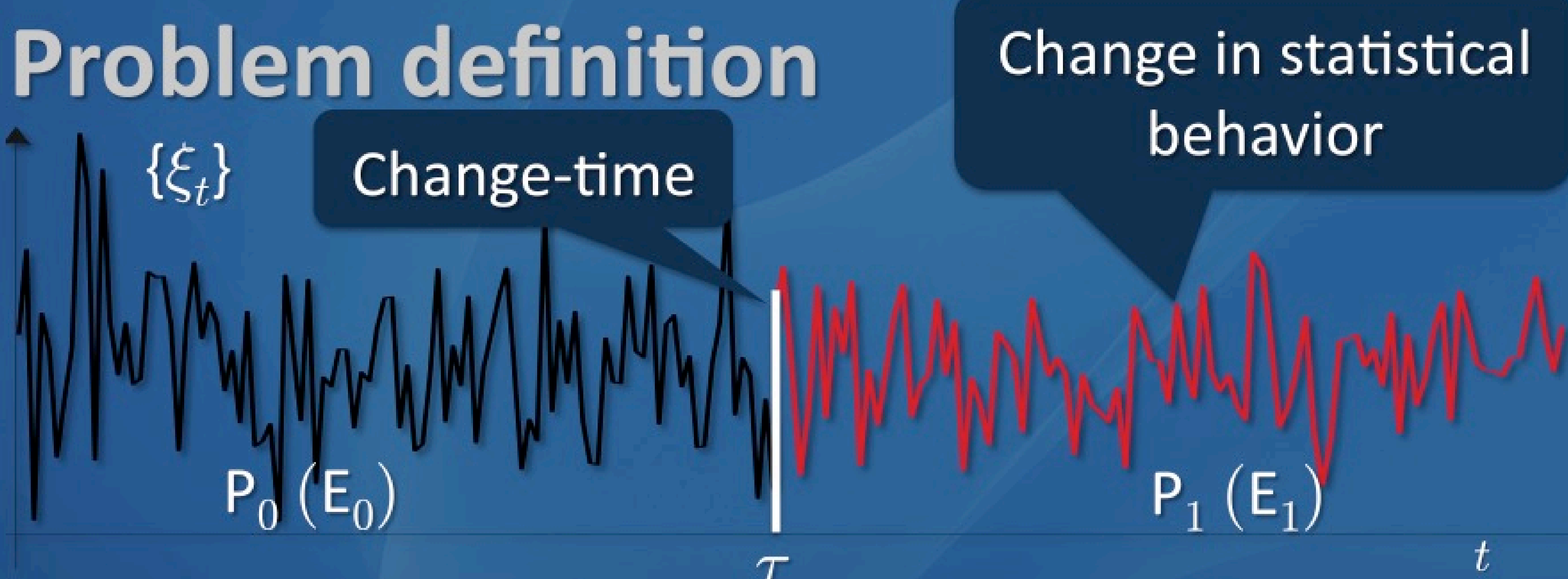
Metrics and Optimum Tests
for Sequential Change-Detection

George V. Moustakides
University of Patras, Greece
& Rutgers University, USA

Outline

- Definition of the change detection problem, Applications, Questions with existing formulations
- Model for mechanism that imposes change
- Two generic setups for sequential detection of changes
  - Complete knowledge
  - Incomplete knowledge

Problem definition



Detect change as soon as possible

Data become available sequentially: at each instant t obtain new sample ξ_t.

Detector: Every instant t consult available data ξ_1 ,..., ξ_t
Use them to make **Binary Decision**

Each instant t decide between:

- ~~A change took place before and including t.~~
- ~~A change didn't take place before and including t.~~

STOP

Ask for more observations

There will be a point in time (call it) T **where we stop**.

T is random time controlled by the observations.

Need a test to implement binary decision at each time t:

$$G_t(\xi_1, \ldots, \xi_t) \underset{\text{Sample}}{\overset{\text{Stop}}{\gtreqless}} \nu$$

Stopping rule

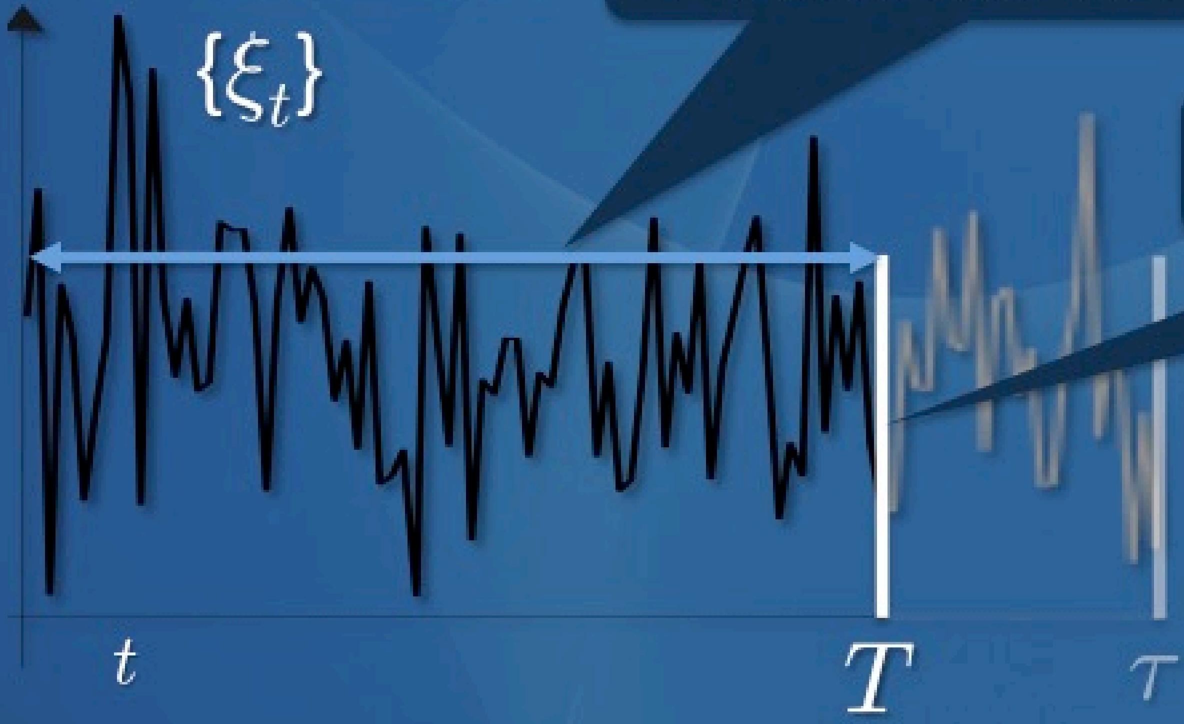
$$T = \inf\{G_t(\xi_1, \ldots, \xi_t) \geq \nu\}$$

Stopping Time *adapted* to the observation history

Class of stopping times: **Extremely rich**

False Alarm

False Alarm Period



Probability of False Alarms

Design T to **minimize** Detection Delay $(T - \tau)$ while **controlling** False Alarms

(Successful) Detection

Detection Delay $T - \tau$

Quality monitoring of manufacturing process

Production line → Continuous Measurements → Quality Assessment

Medical Applications

Epidemic Detection

Disease rate measurements → Increase in rate → Epidemic outbreak?

Early Detection of Epilepsy Episode

EEG Wearables → Divergence from normal → Episode?

Financial Applications

- Structural Change-detection in Exchange Rates
- Portfolio Monitoring

Electronic Communications

Seismology

Speech & Image Processing (segmentation)

Vibration monitoring (Structural health monitoring)

Security monitoring (fraud detection)

Spectrum monitoring

Scene monitoring

Network monitoring (router failures, attack detection)

⋮

CUSUM: 3,000 hits in 2015. Google Scholar.

80% in Change Detection: 2300 articles

Performance Metrics

$$J(T) = \mathsf{E}_1[T - \tau \,|\, T > \tau]$$

What is the change-time τ?

Metric must measure only success

Failures will be dealt through False Alarm control

Shiryaev (1963): τ is random with known prior:

$$J_{\mathsf{S}}(T) = \mathsf{E}_1[T - \tau \,|\, T > \tau]$$

Too restrictive!

Pollak (1985): τ is deterministic and unknown:

$$J_{\mathsf{P}}(T) = \sup_{t \geq 0} \mathsf{E}_1[T - t \,|\, T > t]$$

Leads to SR test

Lorden (1971): τ is deterministic and unknown:

$$J_{\mathsf{L}}(T) = \sup_{t \geq 0} \sup_{\xi_1, \ldots, \xi_t} \mathsf{E}_1[T - t \,|\, T > t, \xi_1, \ldots, \xi_t]$$

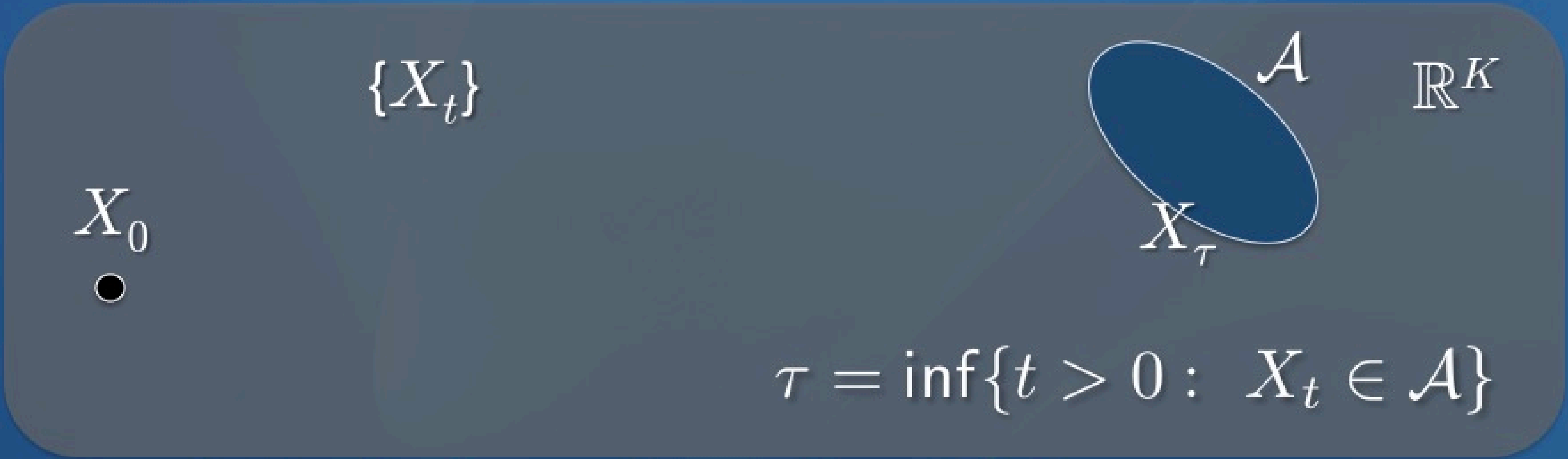
Leads to CUSUM

Too pessimistic (?)

Model for change imposing mechanism

A random vector process $\{X_t\}$ evolves in time in $\mathbb{R}^K$

$\mathcal{A}$ is a subset in $\mathbb{R}^K$



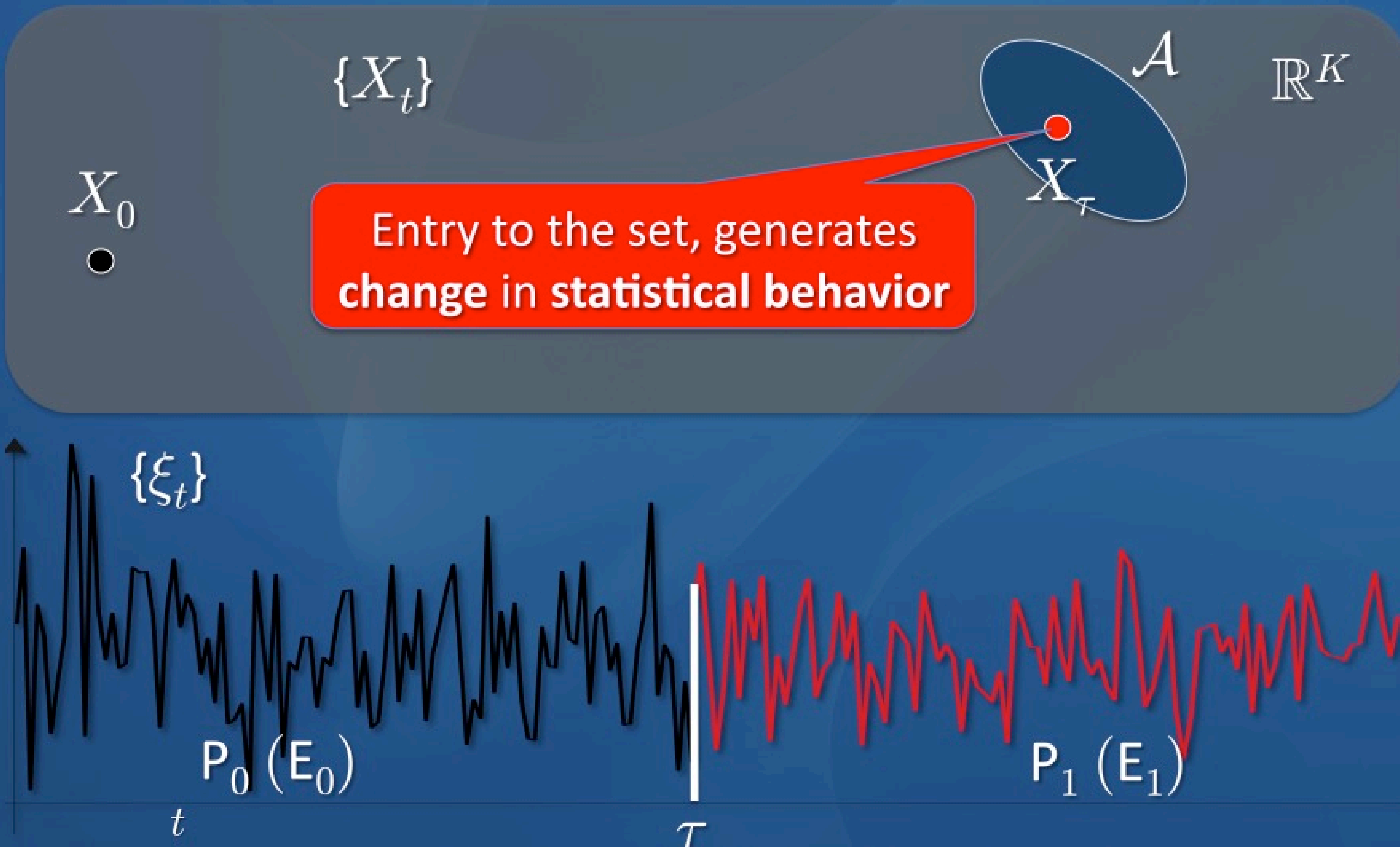
τ is a first entry time, depends on $\{X_t\}$.

We would like to detect it.

If $\{X_t\}$ observable and $\mathcal{A}$ known, problem is trivial.

If $\{X_t\}$ (partially) hidden and/or $\mathcal{A}$ unknown, problem is challenging.

Instead of $\{X_t\}$ we observe process $\{\xi_t\}$



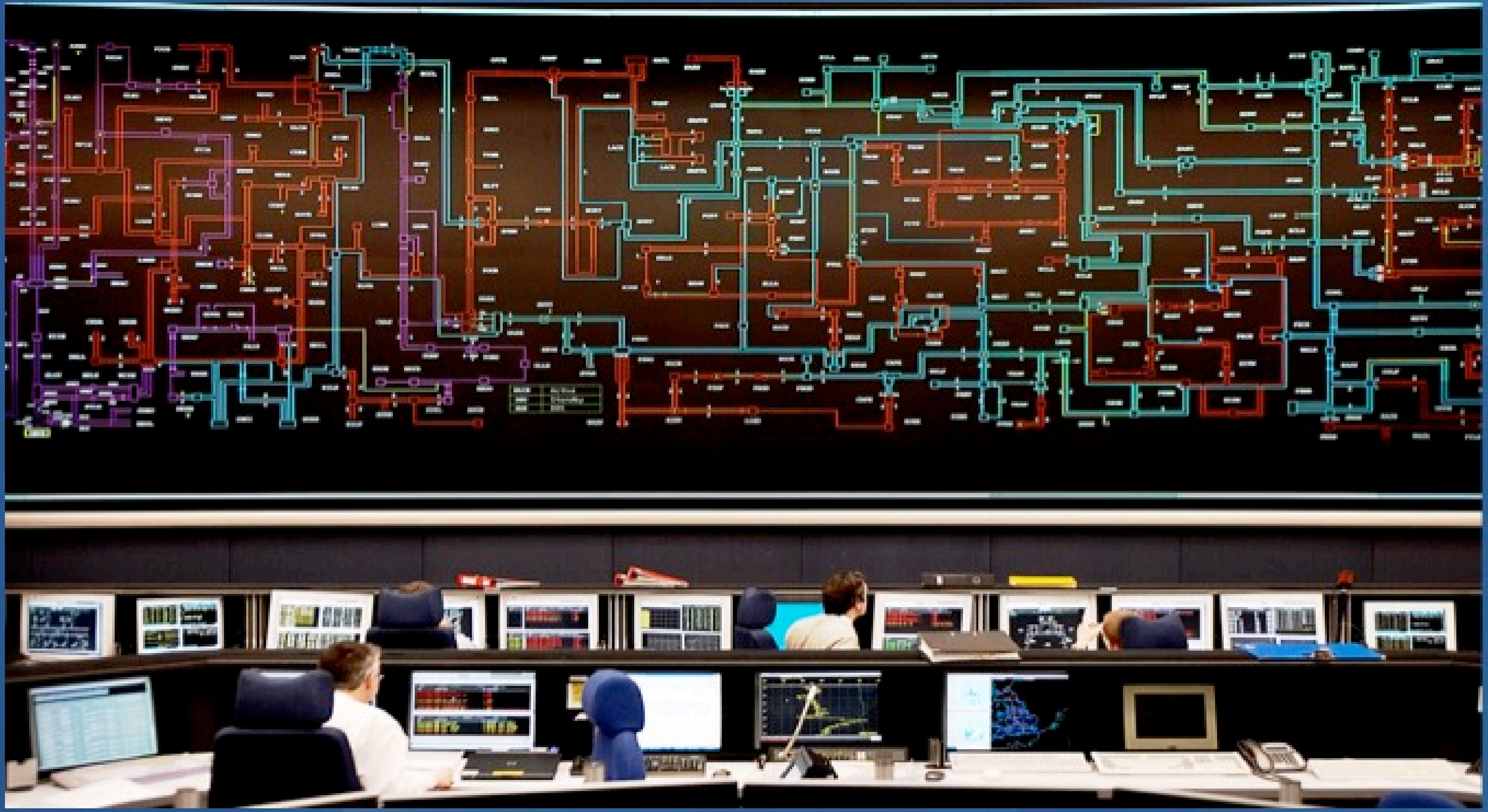
Sequential Change Detection Problem

- First-entry: **Model for change-imposing mechanism.**
- Unifies all existing formulations
- Understanding of change-imposing mechanism can explain existing metrics
- May lead to more efficient detectors.

Goal: detect occurrence of τ

τ is a stopping time adapted to the history generated by the (partially) hidden process $\{X_t\}$.

T is a stopping time adapted to the history generated by the observation sequence $\{\xi_t\}$.



Power Grid:

X_t: Energy at major points in the grid.

$\xi_t = X_t + W_t$ noisy measurements.

$\mathcal{A}$: If $X_t \in \mathcal{A}$ then, after short time major blackout.
$\mathcal{A}$ is known

Structure health monitoring:

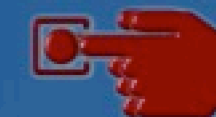
Dependent

X_t: Vibrations at every point of the structure (state)

$\xi_t = \mathbf{A}X_t + W_t$: Low dimensional noisy measurements

$\mathcal{A}$: If $X_t \in \mathcal{A}$ then cracks (**change** in the structure)

$\mathcal{A}$ known or unknown.



Independent {X_t} and {ξ_t} ?:

X_t: Field coordinates of the ball

ξ_t: Noisy vibration measurements

Independent

$\mathcal{A}$: Volume under the goal net.

Performance metrics - known entry set

Delayed Detection

$$\mathcal{J}(T) = \mathsf{E}_1[T - \tau | T > \tau]$$

$$\inf_T \mathcal{J}(T) = \inf_T \mathsf{E}_1[T - \tau | T > \tau]$$

$$\text{subject to : } \mathsf{P}_0(T \leq \tau) \leq \alpha$$

Hard Limited Delay

$$\mathcal{P}(T) = \mathsf{P}_1(T \leq \tau + M | T > \tau)$$

$$\sup_T \mathcal{P}(T) = \sup_T \mathsf{P}_1(T \leq \tau + M | T > \tau)$$

$$\text{subject to : } \mathsf{P}_0(T \leq \tau) \leq \alpha$$

Delayed detection - known entry set

$$\inf_T \mathcal{J}(T) = \inf_T \mathsf{E}_1[T - \tau | T > \tau]$$
$$\text{subject to : } \mathsf{P}_0(T \leq \tau) \leq \alpha$$

Optimal Stopping Theory

Pair process $\{(X_t, \xi_t)\}$ is i.i.d. before and after τ with joint pdfs f_0,f_1.

Moustakides (2016): The optimum test

$$\pi_t = \mathsf{P}_0(X_t \in \mathcal{A} | \xi_t)$$

$$T_o = \inf\{t > 0 : S_t \geq \nu\}$$

$$S_t = S_{t-1} \frac{f_1(\xi_t)}{(1 - \pi_t) f_0(\xi_t)} + \frac{\pi_t}{1 - \pi_t}$$

If additionally $\{X_t\}$ and $\{\xi_t\}$ **independent processes**, then

$$\pi_t = \mathsf{P}_0(X_t \in \mathcal{A}|\xi_t) = \mathsf{P}_0(X_t \in \mathcal{A}) = \pi$$

$$S_t = S_{t-1}\frac{f_1(\xi_t)}{(1-\pi)f_0(\xi_t)} + \frac{\pi}{1-\pi}$$

$$\tilde{S}_t = \frac{1-\pi}{\pi}S_t - 1, \quad \tilde{\nu} = \frac{1-\pi}{\pi}\nu - 1$$

$$T_o = \mathsf{inf}\{t > 0 : \ \tilde{S}_t \geq \tilde{\nu}\}$$

$$\tilde{S}_t = (\tilde{S}_{t-1} + 1)\frac{f_1(\xi_t)}{(1-\pi)f_0(\xi_t)}$$

Shiryayev test (1963)

Unknown entry set

- What if entry set $\mathcal{A}$ is unknown?
- Can detect the first-entry to an unknown set? Equivalently: can detect the change-time τ that inflicts a change in the statistical behavior?

Focus on
change of the statistics

Performance metrics - unknown entry set

Delayed Detection

Worst-case analysis

$$\mathcal{J}(T) = \sup_{\tau} \mathsf{E}_1[T-\tau | T>\tau]$$

$$\inf_{T} \mathcal{J}(T) = \inf_{T} \sup_{\tau} \mathsf{E}_1[T-\tau | T>\tau]$$

$$\text{subject to}: \mathsf{E}_0[T] \geq \gamma$$

Hard Limited Delay

$$\mathcal{P}(T) = \inf_{\tau} \mathsf{P}_1(T \leq \tau + M | T>\tau)$$

$$\sup_{T} \mathcal{P}(T) = \sup_{T} \inf_{\tau} \mathsf{P}_1(T \leq \tau + M | T>\tau)$$

$$\text{subject to}: \mathsf{E}_0[T] \geq \gamma$$

Delayed detection - unknown entry set

$$\mathcal{J}(T) = \sup_{\tau} \mathsf{E}_1[T-\tau | T>\tau]$$

We can show (Moustakides 2008):

$\{X_t\}$ **and** $\{\xi_t\}$ **independent processes**

$$\mathcal{J}_{\mathsf{P}}(T) = \sup_{t\geq 0} \mathsf{E}_1[T-t | T>t]$$ Pollak (1985)

$\{X_t\}$ **and** $\{\xi_t\}$ **dependent processes**

$$\mathcal{J}_{\mathsf{L}}(T) = \sup_{t\geq 0} \sup_{\xi_1,\ldots,\xi_t} \mathsf{E}_1[T-t | T>t, \xi_1, \ldots, \xi_t]$$

Lorden (1971)

$$\mathcal{J}_{\mathsf{P}}(T) = \sup_{t\geq 0} \mathsf{E}_1[T-t|T>t] \quad \text{Pollak (1985)}$$

$$\inf_T J_{\mathsf{P}}(T) \;\; \text{subject to} : \mathsf{E}_0[T] \geq \gamma$$

Discrete time: i.i.d. data before and after the change with pdfs f_0,f_1.

Shiryaev-Roberts-Pollak test

Compute recursively the following statistic:

$$S_t = (S_{t-1}+1)\frac{f_1(\xi_t)}{f_0(\xi_t)} \quad T_{\mathsf{P}} = \inf\{t>0 : S_t \geq \nu\}$$

Pollak (1985): If S_0 specially designed, then

$$[J_{\mathsf{P}}(T_{\mathsf{P}}) - \inf_T J_{\mathsf{P}}(T)] \to 0; \quad \text{as } \gamma \to \infty$$

Exact optimality?

~~Yakir (1997)~~ Mei (2007)

Polunchenko-Tartakovsky (2012) counterexample.

Change in the drift of a BM: Polunchenko (2016)

Dependence? Multiple pre- and/or post-change possibilities? Time variation?

$$\mathcal{J}_{\mathsf{L}}(T) = \sup_{t\geq 0} \sup_{\xi_1,\ldots,\xi_t} \mathsf{E}_1[T-t|T>t,\xi_1,\ldots,\xi_t]$$

Lorden (1971)

$$\inf_T J_{\mathsf{L}}(T) \ \text{ subject to : } \mathsf{E}_0[T] \geq \gamma$$

Discrete time: i.i.d. data before and after the change with pdfs f_0,f_1.

CUSUM test

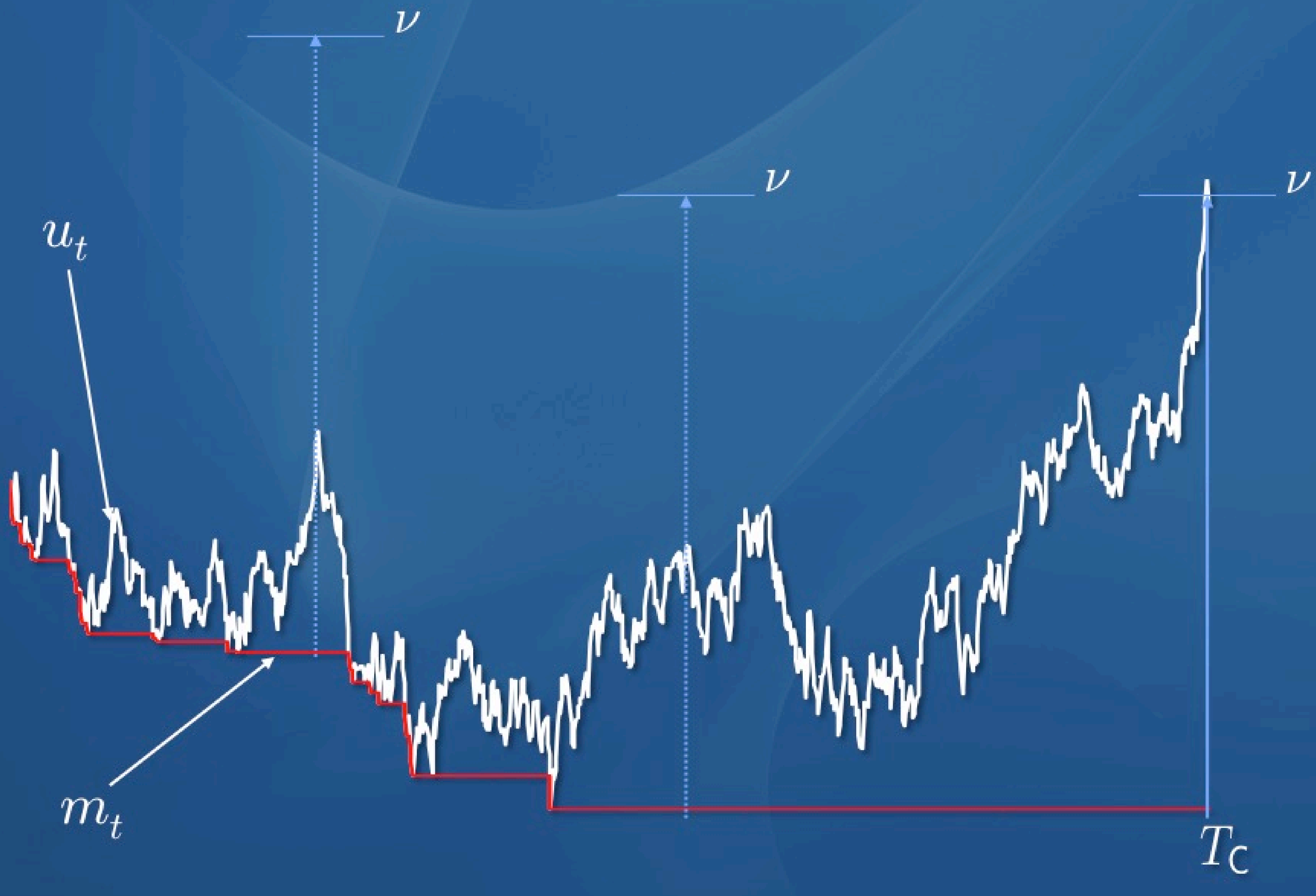
$$S_t = \max\{S_{t-1},0\} + \log\frac{f_1(\xi_t)}{f_0(\xi_t)}$$

$$T_{\mathsf{C}} = \inf\{t>0 : S_t \geq \nu\}$$

$$u_t = \sum_{s=1}^{t} \log\frac{f_1(\xi_s)}{f_0(\xi_s)}$$

$$m_t = \min_{0\leq s\leq t} u_s$$

$$S_t = u_t - m_t$$



ν
ν
ν
u_t
m_t
T_C

Discrete time: Lorden (1971) asymptotic optimality.
Moustakides (1986) exact optimality.

Continuous time: Shiryaev (1996), Beibel (1996)
Brownian Motion with constant drift before and after.
Moustakides (2004) strict optimality for Ito processes.

Discrete time: Moustakides-Veeravalli (2016) Non abrupt changes

Dependence? Multiple pre- and/or post-change possibilities?

Hard Limited Delay

$$\mathsf{P}_1(T \le \tau + M | T > \tau)$$

Only for $M = 1$: $\mathsf{P}_1(T = \tau + 1 | T > \tau)$

Corresponds to immediate detection with the first sample after the change

$$\mathcal{P}_{\mathsf{S}}(T) = \mathsf{P}_1(T = \tau + 1 | T > \tau)$$ Shiryaev like

$$\mathcal{P}_{\mathsf{P}}(T) = \inf_{t \ge 0} \mathsf{P}_1(T = t + 1 | T > t)$$ Pollak like

$$\mathcal{P}_{\mathsf{L}}(T) = \inf_{t \ge 0} \inf_{\xi_1, \ldots, \xi_t} \mathsf{P}_1(T = t + 1 | T > t, \xi_1, \ldots, \xi_t)$$ Lorden like

$$\sup_T \mathcal{P}_{\mathsf{S}}(T)$$
$$\text{s.t.}: \mathsf{P}_0(T \le \tau) \le \alpha$$

$$\sup_T \mathcal{P}_{\mathsf{P(L)}}(T)$$
$$\text{s.t.}: \mathsf{E}_0[T] \ge \gamma$$

$$T_{\mathsf{Sh}} = \inf\left\{t > 0 : \frac{f_1(\xi_t)}{f_0(\xi_t)} \geq \nu\right\}$$

Shewhart test (1931)

Optimality: Bojdecki (1979): Shiryaev like

Pollak and Krieger (2013): Pollak like

Moustakides (2014): Lorden like

Pollak and Krieger (2013): Multiple post-change possibilities.

Moustakides (2014): Post change time variation

Dependent observations

$$\mathcal{P}_{\mathsf{L}}(T) = \inf_{t\geq 0} \inf_{\xi_1,\ldots,\xi_t} \mathsf{P}_1(T = t+1 | T > t, \xi_1, \ldots, \xi_t)$$

$$\sup_T \mathcal{P}_{\mathsf{L}}(T) \quad \text{subject to : } \mathsf{E}_0[T] \geq \gamma$$

Markovian pre- and post-change observations $\{\xi_t\}$

$$T_{\mathsf{Sh}} = \inf\left\{t > 0 : c(\xi_{t-1}) \frac{f_1(\xi_t | \xi_{t-1})}{f_0(\xi_t | \xi_{t-1})} \geq \nu(\xi_t)\right\}$$

Moustakides (2015): With properly designed functions $c(\xi)$ and $\nu(\xi)$ we solve the constrained optimization.

Simple solution for conditionally Gaussian pdfs.

$$\mathcal{J}_{\mathsf{L}}(T) = \sup_{t\geq 0} \sup_{\xi_1,\ldots,\xi_t} \mathsf{E}_1[T-t|T>t,\xi_1,\ldots,\xi_t]$$

$$\inf_T J_{\mathsf{L}}(T) \ \text{ subject to : } \mathsf{E}_0[T] \geq \gamma$$

For Markovian pre- and post-change $\{\xi_t\}$: Solution ??

Acknowledgement

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Thank you for your attention!