Sequential Detection of Changes

Metrics and Optimum Tests

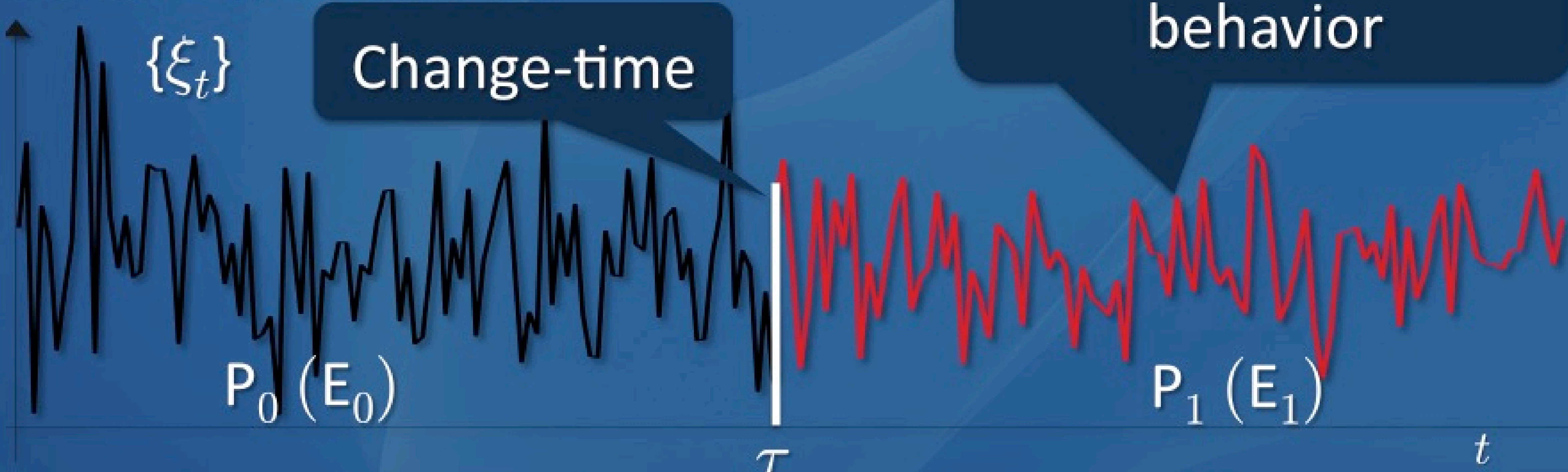
George V. Moustakides

University of Patras, Greece

Outline

- Introduction: Problem definition, Applications, Existing formulations - Questions
- A model for the change imposing mechanism
- Two generic setups for sequential change detection
  - Complete knowledge - Shiryaev's metric
  - Pollak's metric
  - Lorden's metric

Introduction



Detect change as soon as possible

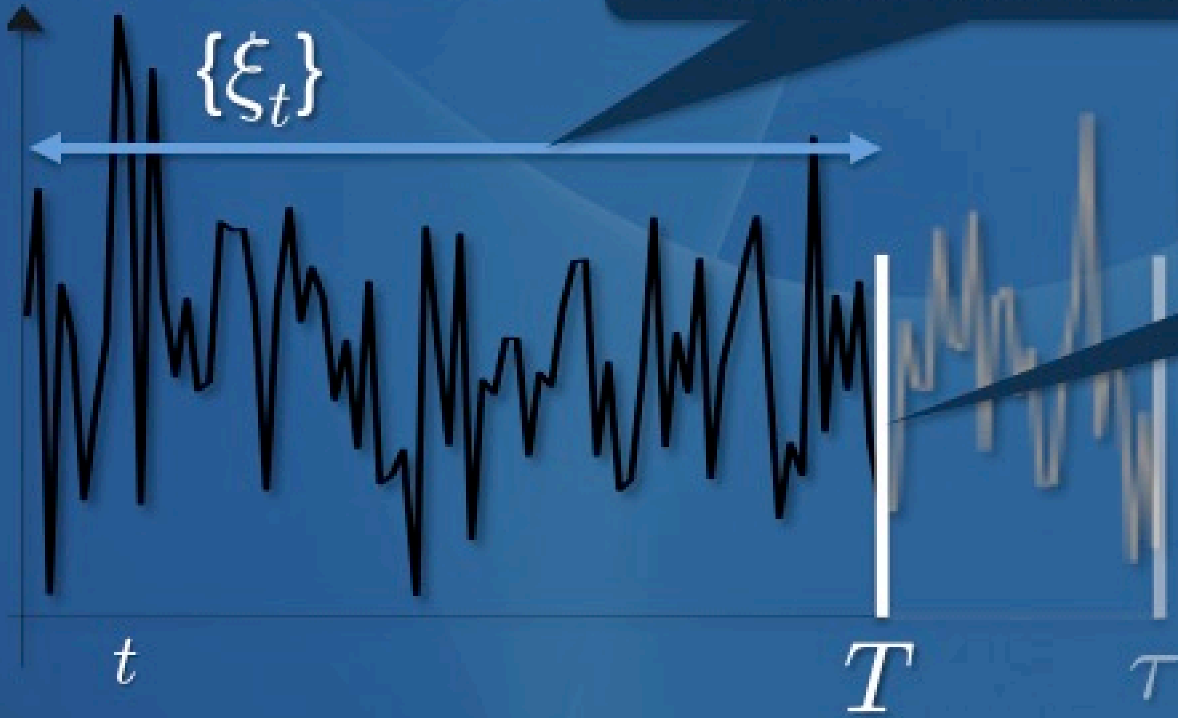
Data become available sequentially: at each time instant t we obtain a new sample ξ_t.

At every time t consult available data ξ_1 ,..., ξ_t and **decide**

- ~~A change took place:~~ **STOP** **Stopping Time** T
- ~~A change didn't take place:~~ Ask for next sample ξ_{t+1}

False Alarm

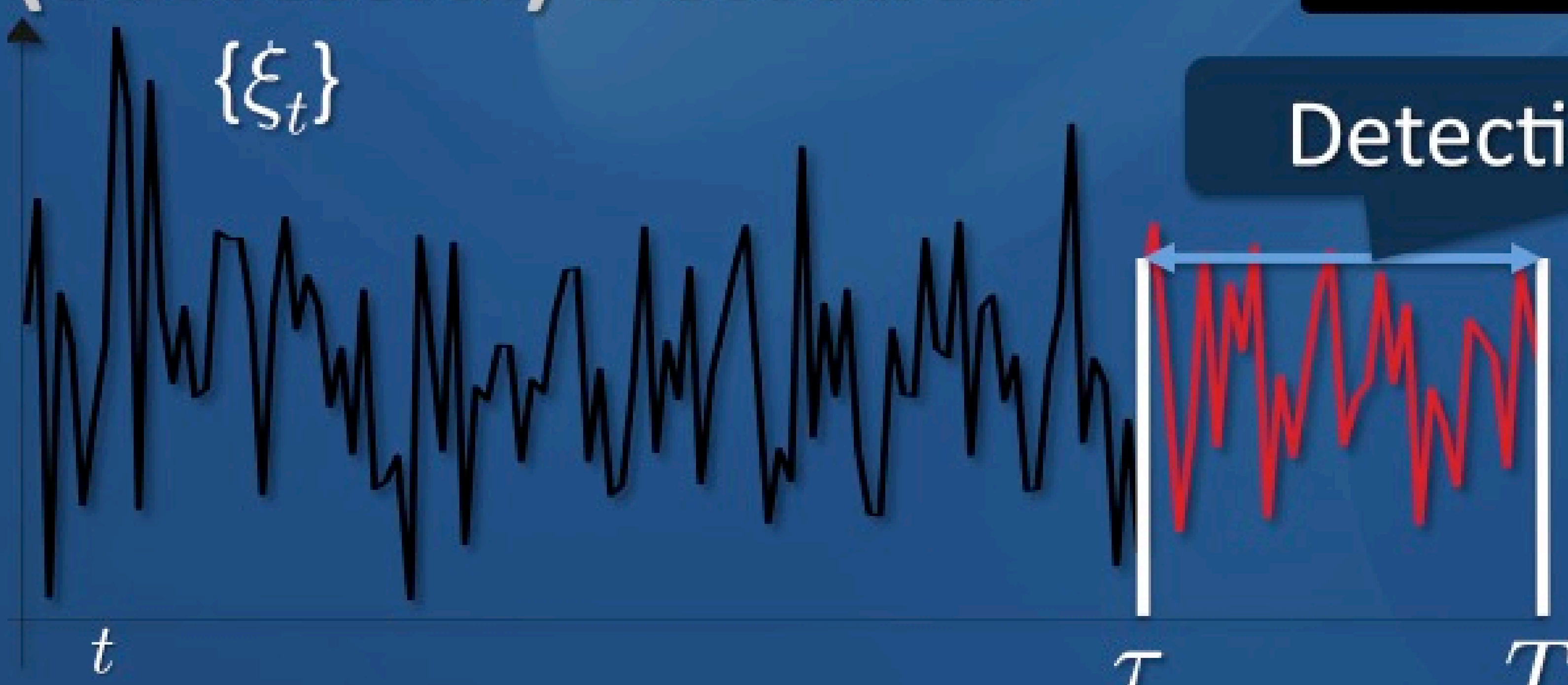
False Alarm Period



Probability of False Alarms

Design T to **minimize** Detection Delay $(T - \tau)$ while **controlling** False Alarms

(Successful) Detection



Detection Delay $T - \tau$

Quality monitoring of manufacturing process

Production line → Continuous Measurements → Quality Assessment

Medical Applications

Epidemic Detection

Disease rate measurements → Increase in rate? → Epidemic outbreak?

Early Detection of Epilepsy Episode

EEG Wearables → Divergence from normal → Episode?

Financial Applications

- Structural Change-detection in Exchange Rates
- Portfolio Monitoring

Electronic Communications

Seismology

Speech & Image Processing (segmentation)

Vibration monitoring (Structural health monitoring)

Security monitoring (fraud detection)

Spectrum monitoring

Scene monitoring

Network monitoring (router failures, attack detection)

⋮

CUSUM: 3,000 hits in 2015. Google Scholar.

80% in Change Detection: 2300 articles

Performance Metrics

$$J(T) = \mathsf{E}_1[T - \tau \,|\, T > \tau]$$

What is the change-time τ ?

Metric must measure only success

Failures will be dealt through False Alarm control

Shiryaev (1963): τ is random with known prior:

$$J_{\mathsf{S}}(T) = \mathsf{E}_1[T - \tau \,|\, T > \tau]$$

Too restrictive!

Pollak (1985): τ is deterministic and unknown:

$$J_{\mathsf{P}}(T) = \sup_{t \geq 0} \mathsf{E}_1[T - t \,|\, T > t]$$

Leads to SR test

Lorden (1971): τ is deterministic and unknown:

$$J_{\mathsf{L}}(T) = \sup_{t \geq 0} \sup_{\xi_1, \ldots, \xi_t} \mathsf{E}_1[T - t \,|\, T > t, \xi_1, \ldots, \xi_t]$$

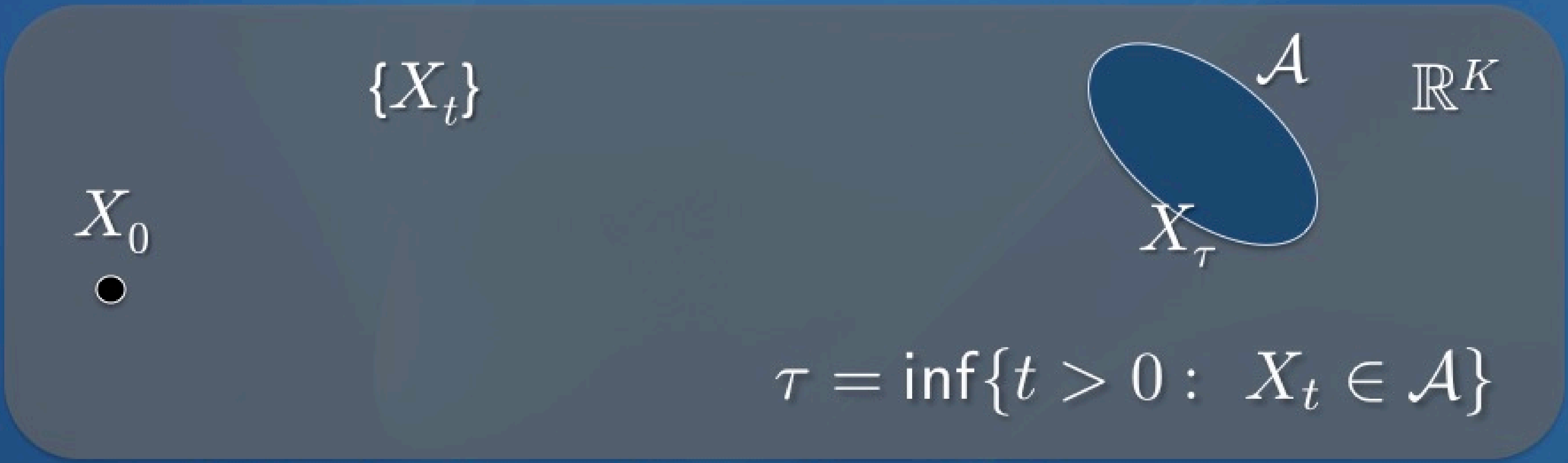
Leads to CUSUM

Too pessimistic (?)

Model for change imposing mechanism

A random vector process $\{X_t\}$ evolves in time in $\mathbb{R}^K$

$\mathcal{A}$ is a subset in $\mathbb{R}^K$



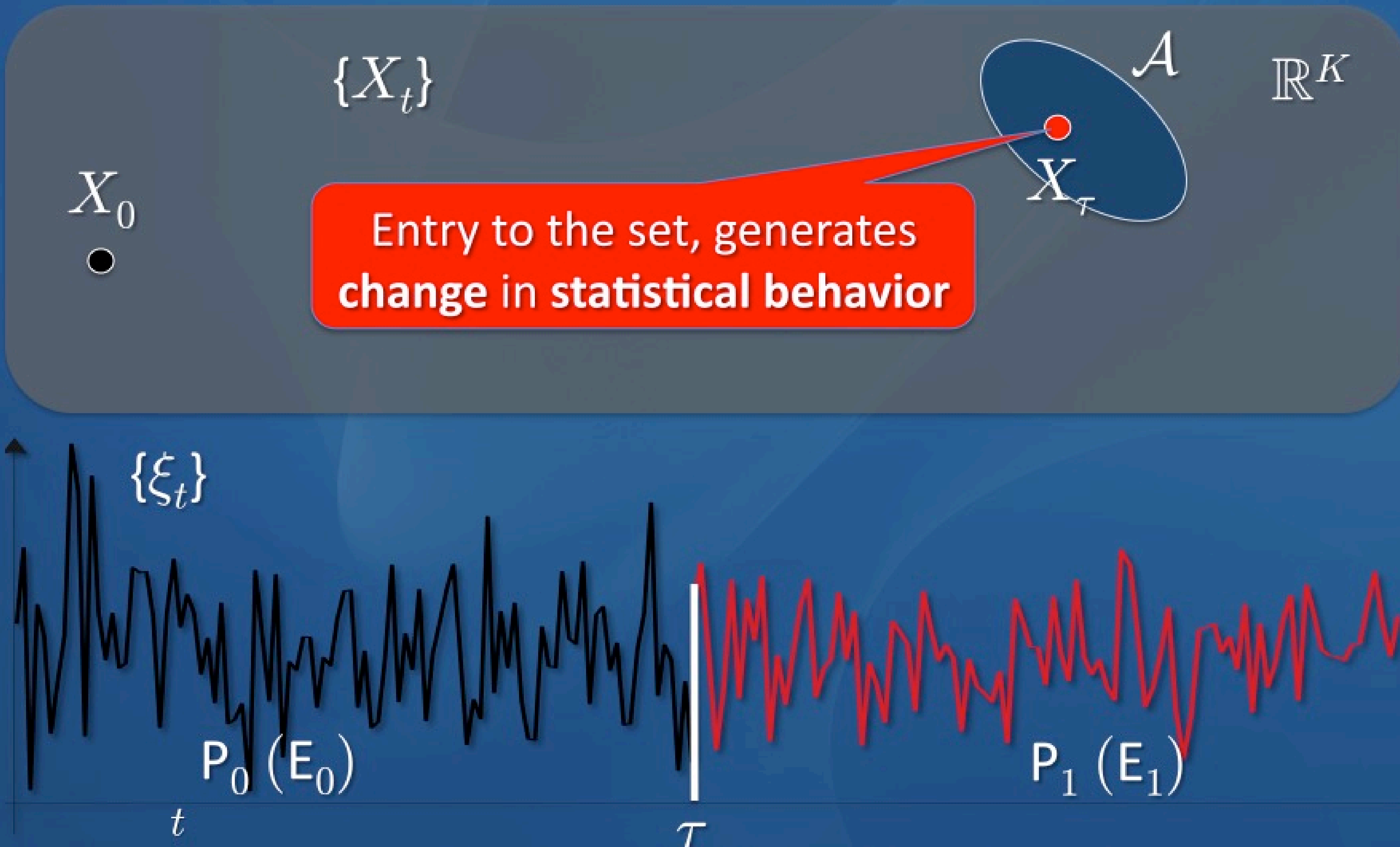
τ is a first entry time, depends on $\{X_t\}$.

We would like to detect it.

If $\{X_t\}$ observable and $\mathcal{A}$ known, problem is trivial.

If $\{X_t\}$ (partially) hidden and/or $\mathcal{A}$ unknown, problem is challenging.

Instead of $\{X_t\}$ we observe process $\{\xi_t\}$



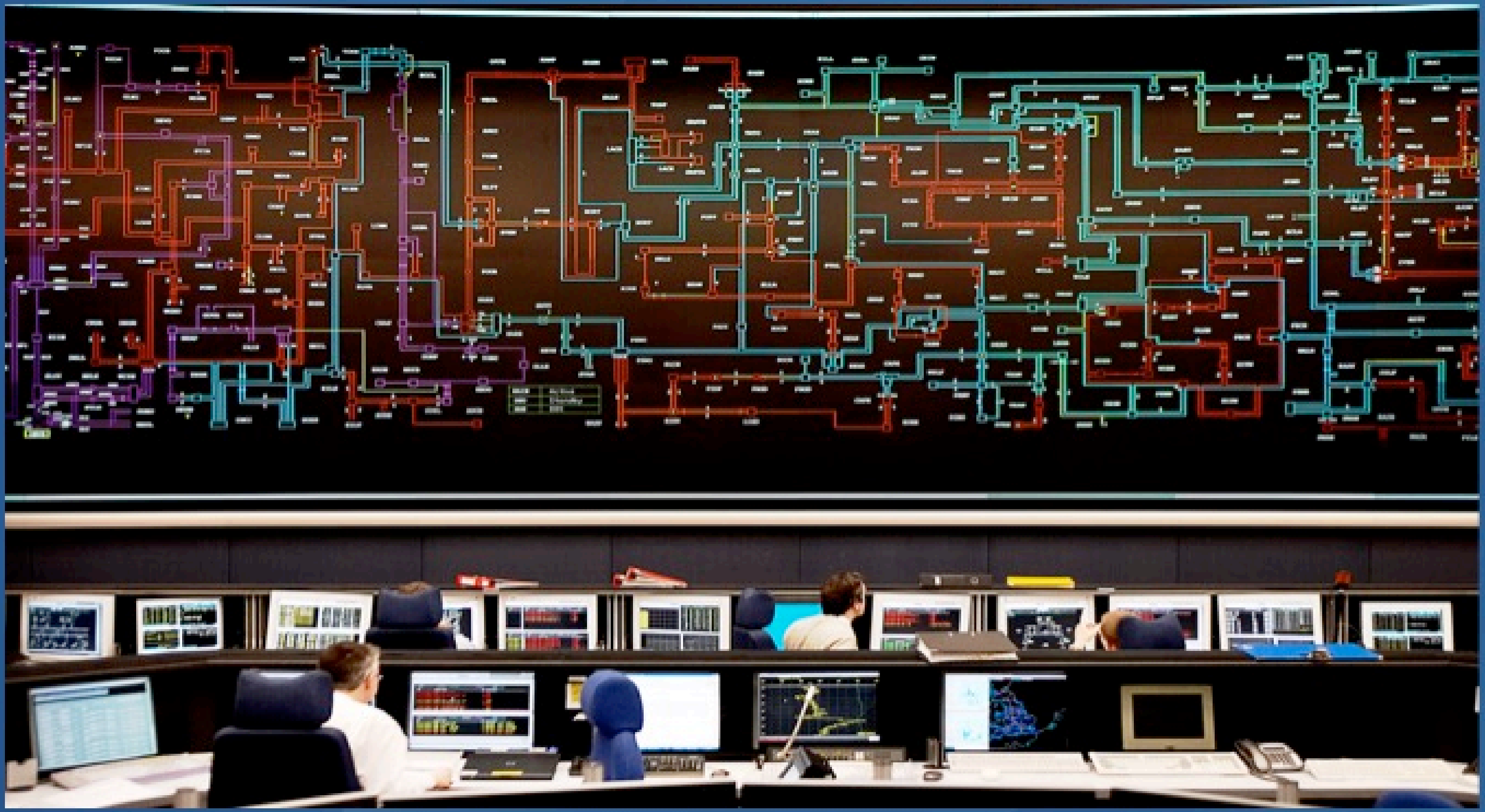
Sequential Change Detection Problem

- First-entry: **Model for change-imposing mechanism.**
- Unifies all existing formulations
- Deeper understanding of change-imposing mechanism may lead to more efficient detectors.

Goal: detect occurrence of τ

τ is a stopping time adapted to the filtration generated by the (partially) hidden process $\{X_t\}$.

T is a stopping time adapted to the filtration generated by the observation sequence $\{\xi_t\}$.



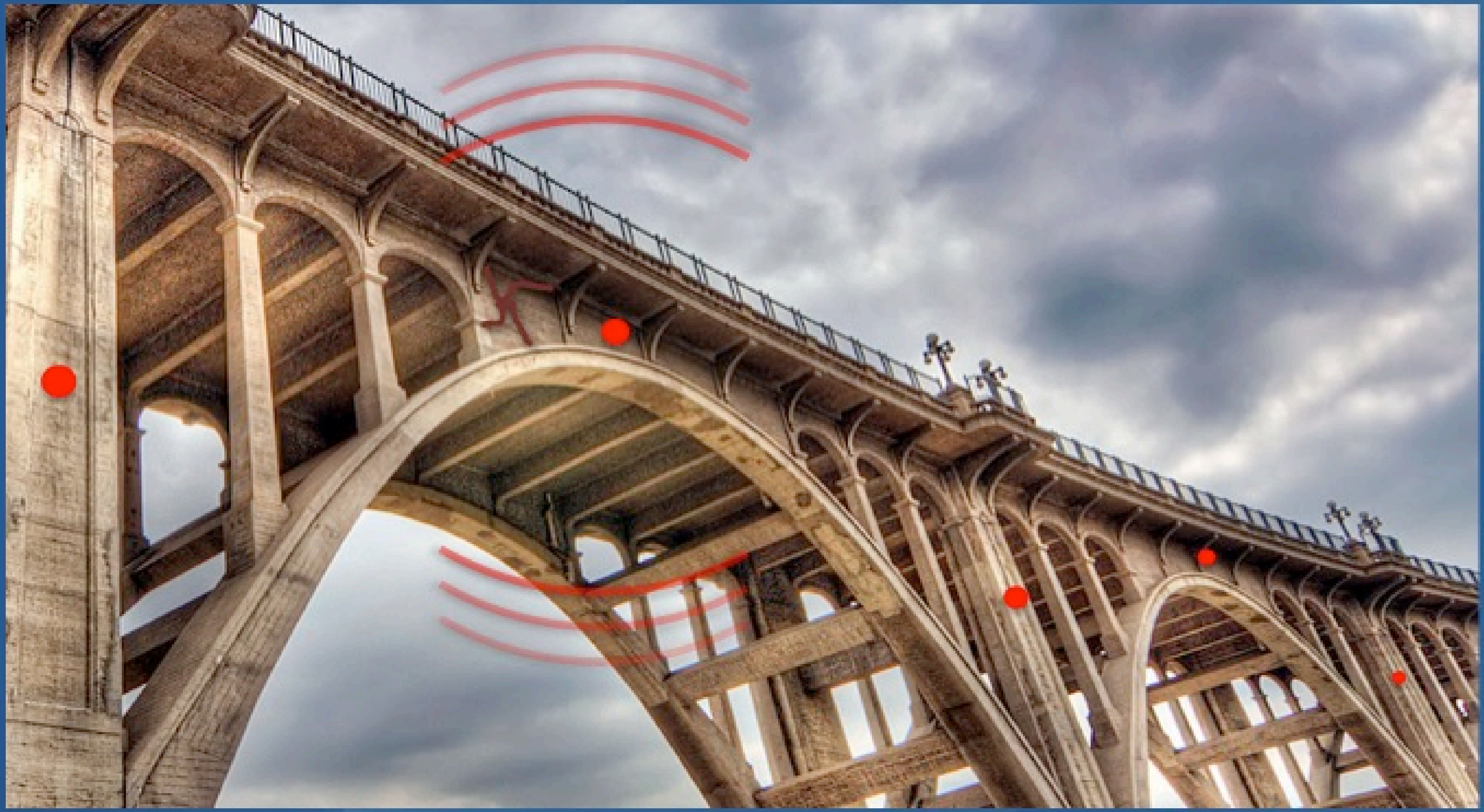
Power Grid:

X_t: Energy at major points in the grid.

$\xi_t = X_t + W_t$ noisy measurements.

$\mathcal{A}$: If $X_t \in \mathcal{A}$ then, after short time major blackout.

$\mathcal{A}$ is known



Structure health monitoring:

X_t: Vibrations at every point of the structure (state)

$\xi_t = \mathbf{A}X_t + W_t$: Low dimensional noisy measurements

$\mathcal{A}$: If $X_t \in \mathcal{A}$ then cracks (**change** in the structure)

$\mathcal{A}$ known or unknown.

Independent $\{X_t\}$ and $\{\xi_t\}$:

X_t: Field coordinates of the ball

ξ_t: Noisy measurements

Independent

$\mathcal{A}$: Volume under the goal net.

Performance metrics - known entry set

Delayed Detection (Tracking)

$$\mathcal{J}(T) = \mathsf{E}_1[T - \tau | T > \tau]$$

$$\inf_T \mathcal{J}(T) = \inf_T \mathsf{E}_1[T - \tau | T > \tau]$$

$$\text{subject to : } \mathsf{P}_0(T \leq \tau) \leq \alpha$$

Hard Limited Delay (Capture)

$$\mathcal{P}(T) = \mathsf{P}_1(T \leq \tau + M | T > \tau)$$

$$\sup_T \mathcal{P}(T) = \sup_T \mathsf{P}_1(T \leq \tau + M | T > \tau)$$

$$\text{subject to : } \mathsf{P}_0(T \leq \tau) \leq \alpha$$

Delayed detection - known entry set

$$\inf_T \mathcal{J}(T) = \inf_T \mathsf{E}_1[T - \tau | T > \tau]$$
$$\text{subject to}: \mathsf{P}_0(T \le \tau) \le \alpha$$

Optimal Stopping Theory

Pair process $\{(X_t, \xi_t)\}$ is i.i.d. before and after τ with joint pdfs f_0, f_1.

Moustakides (2016): The optimum test

$$\pi_t = \mathsf{P}_0(X_t \in \mathcal{A} | \xi_t)$$

$$T_o = \inf\{t > 0 : S_t \ge \nu\}$$

$$S_t = S_{t-1} \frac{f_1(\xi_t)}{(1 - \pi_t) f_0(\xi_t)} + \frac{\pi_t}{1 - \pi_t}$$

If additionally $\{X_t\}$ and $\{\xi_t\}$ **independent processes**, then

$$\pi_t = \mathsf{P}_0(X_t \in \mathcal{A}|\xi_t) = \mathsf{P}_0(X_t \in \mathcal{A}) = \pi$$

$$S_t = S_{t-1}\frac{f_1(\xi_t)}{(1-\pi)f_0(\xi_t)} + \frac{\pi}{1-\pi}$$

$$\tilde{S}_t = \frac{1-\pi}{\pi}S_t - 1, \quad \tilde{\nu} = \frac{1-\pi}{\pi}\nu - 1$$

$$\tilde{S}_t = (\tilde{S}_{t-1}+1)\frac{f_1(\xi_t)}{(1-\pi)f_0(\xi_t)}$$

$$T_o = \inf\{t > 0: \ \tilde{S}_t \geq \tilde{\nu}\}$$

Shiryayev test (1963)

Unknown entry set

- What if entry set $\mathcal{A}$ is unknown?
- Can we still detect the first-entry to an unknown set? Equivalently detect the change-time τ that inflicts change in statistics?

Focus on
change of the statistics

Performance metrics - unknown entry set

Delayed Detection

Worst-case analysis

$$\mathcal{J}(T) = \sup_{\tau} \mathsf{E}_1[T-\tau | T>\tau]$$

$$\inf_T \mathcal{J}(T) = \inf_T \sup_{\tau} \mathsf{E}_1[T-\tau | T>\tau]$$

$$\text{subject to}: \mathsf{E}_0[T] \geq \gamma$$

Hard Limited Delay

$$\mathcal{P}(T) = \inf_{\tau} \mathsf{P}_1(T \leq \tau + M | T>\tau)$$

$$\sup_T \mathcal{P}(T) = \sup_T \inf_{\tau} \mathsf{P}_1(T \leq \tau + M | T>\tau)$$

$$\text{subject to}: \mathsf{E}_0[T] \geq \gamma$$

Delayed detection - unknown entry set

$\{X_t\}$ **and** $\{\xi_t\}$ **independent processes**

$$\mathcal{J}_{\mathsf{P}}(T) = \sup_{\tau} \mathsf{E}_1[T-\tau | T>\tau]$$

We can show (Moustakides 2008):

$$\mathcal{J}_{\mathsf{P}}(T) = \sup_{t\geq 0} \mathsf{E}_1[T-t | T>t] \quad \text{Pollak (1985)}$$

$$\inf_T J_{\mathsf{P}}(T) \ \text{ subject to : } \mathsf{E}_0[T] \geq \gamma$$

Discrete time: i.i.d. data before and after the change with pdfs f_0,f_1.

Shiryaev-Roberts-Pollak test

Compute recursively the following statistic:

$$S_t = (S_{t-1} + 1)\frac{f_1(\xi_t)}{f_0(\xi_t)} \quad T_{\mathsf{P}} = \inf\{t > 0 : S_t \geq \nu\}$$

Pollak (1985): If S_0 specially designed, then

$$[J_{\mathsf{P}}(T_{\mathsf{P}}) - \inf_T J_{\mathsf{P}}(T)] \to 0; \quad \text{as } \gamma \to \infty$$

Exact optimality? Tartakovsky (2012) counterexample.

Change in the drift of a BM: Polunchenko (2016)

Dependence? Multiple pre- and/or post-change possibilities? Time variation?

$\{X_t\}$ **and** $\{\xi_t\}$ **dependent processes**

$$\mathcal{J}_{\mathsf{L}}(T) = \sup_{\tau} \mathsf{E}_1[T-\tau|T>\tau]$$

We can show (Moustakides 2008):

$$\mathcal{J}_{\mathsf{L}}(T) = \sup_{t\geq 0} \sup_{\xi_1,\ldots,\xi_t} \mathsf{E}_1[T-t|T>t,\xi_1,\ldots,\xi_t]$$

Lorden (1971)

$$\inf_{T} J_{\mathsf{L}}(T) \text{ subject to : } \mathsf{E}_0[T] \geq \gamma$$

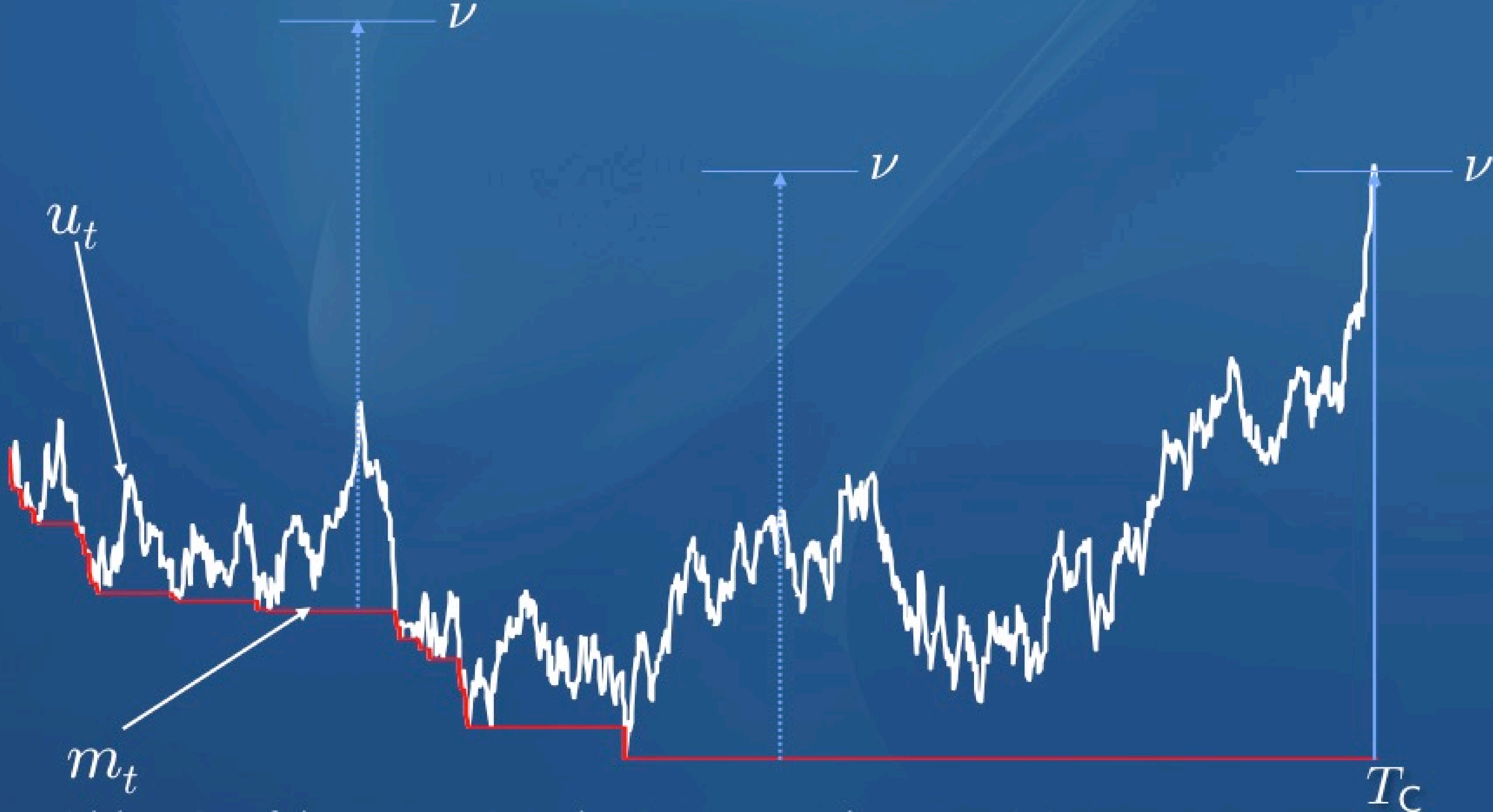
Discrete time: i.i.d. data before and after the change with pdfs f_0,f_1.

CUSUM test

$$S_t = \max\{S_{t-1}, 0\} + \log \frac{f_1(\xi_t)}{f_0(\xi_t)}$$

$$S_t = u_t - m_t$$

$$T_\mathsf{C} = \inf\{t > 0 : S_t \geq \nu\}$$



Discrete time: Lorden (1971) asymptotic optimality. Moustakides (1986) exact optimality.

Continuous time: Shiryaev (1996), Beibel (1996) Brownian Motion with constant drift before and after. Moustakides (2004) strict optimality for Ito processes.

Discrete time: Moustakides-Veeravalli (2016) Non abrupt changes

Dependence? Multiple pre- and/or post-change possibilities?

Hard Limited Delay - unknown entry set

$M=1$ corresponds to immediate detection with the first sample after the change

$$\mathcal{P}_{\mathsf{S}}(T)=\mathsf{P}_1(T=\tau+1|T>\tau)$$

$$\mathcal{P}_{\mathsf{P}}(T)=\inf_{t\geq0}\mathsf{P}_1(T=t+1|T>t)$$

$$\mathcal{P}_{\mathsf{L}}(T)=\inf_{t\geq0}\inf_{\xi_1,\ldots,\xi_t}\mathsf{P}_1(T=t+1|T>t,\xi_1,\ldots,\xi_t)$$

$$\sup_T \mathcal{P}_{\mathsf{S}}(T)$$
$$\text{s.t.}:\mathsf{P}_0(T\leq\tau)\leq\alpha$$

$$\sup_T \mathcal{P}_{\mathsf{P(L)}}(T)$$
$$\text{s.t.}:\mathsf{E}_0[T]\geq\gamma$$

$$T_{\mathsf{Sh}}=\inf\left\{t>0:\frac{f_1(\xi_t)}{f_0(\xi_t)}\geq\nu\right\}$$

Shewhart test (1931)

Optimality: Bojdecki (1979): Shiryaev like

Pollak and Krieger (2013): Pollak like

Moustakides (2014): Lorden like

Pollak and Krieger (2013): Multiple post-change possibilities.

Moustakides (2014): Time variation

Dependent samples

$$\mathcal{P}_{\mathsf{L}}(T) = \inf_{t \geq 0} \inf_{\xi_1,\ldots,\xi_t} \mathsf{P}_1(T = t+1 | T > t, \xi_1, \ldots, \xi_t)$$

$$\sup_{T} \mathcal{P}_{\mathsf{L}}(T) \quad \text{subject to}: \ \mathsf{E}_0[T] \geq \gamma$$

Markovian pre- and post-change pdfs for observations $\{\xi_t\}$

$$T_{\mathsf{Sh}} = \inf\left\{t > 0 : c(\xi_{t-1}) \frac{f_1(\xi_t | \xi_{t-1})}{f_0(\xi_t | \xi_{t-1})} \geq \nu(\xi_t)\right\}$$

Moustakides (2015): With properly designed functions $c(\xi)$ and $\nu(\xi)$ we solve the constrained optimization.

Simple solution for conditionally Gaussian pdfs.