Metrics and Optimum Tests
in Sequential Detection of Changes

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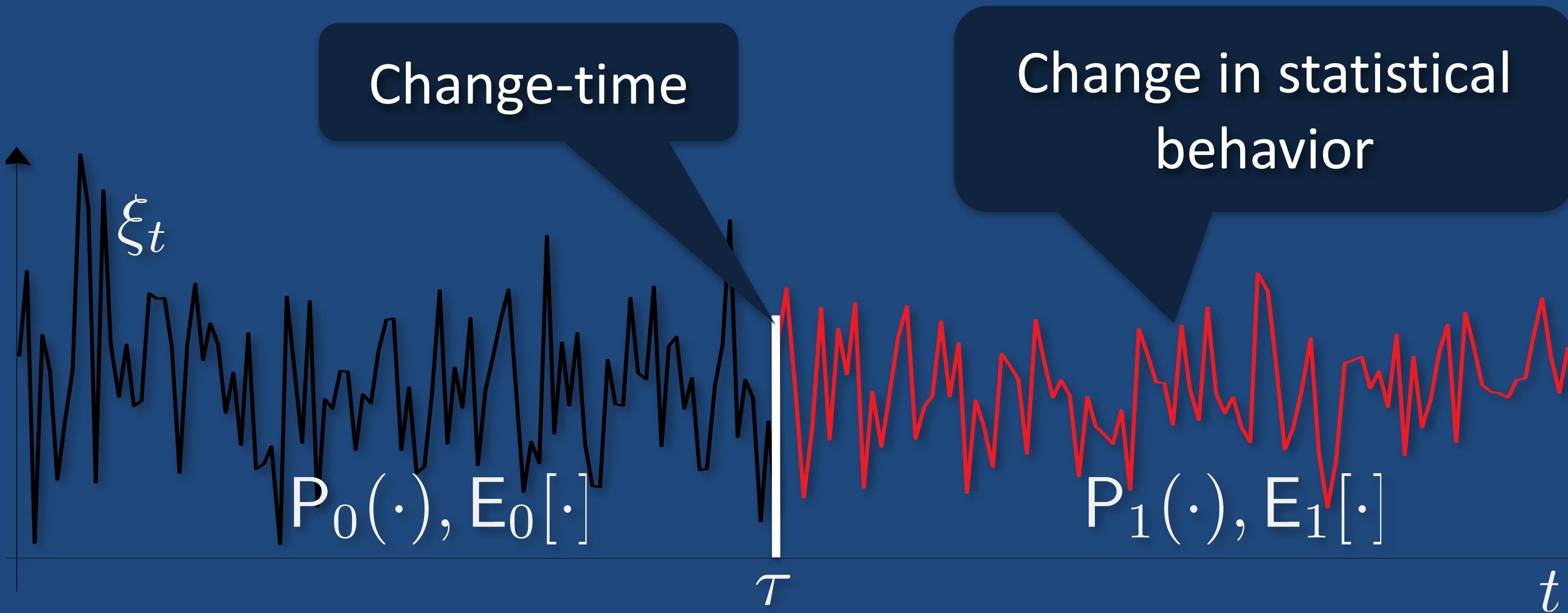
Outline

- Problem definition
- Model for change mechanism
- Performance metrics
- Optimum detectors
- Unknown statistics



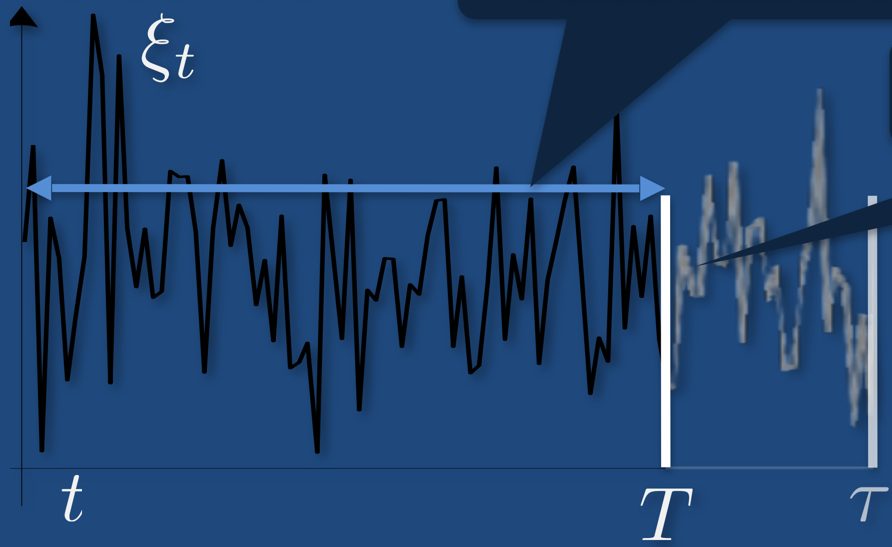
Detect change as soon as possible

Data become available sequentially: at each time t obtain new sample ξ_t

Detector: Each instant t consults available data $\{\xi_t, \ldots, \xi_1\}$ to decide to stop or continue sampling

Detector is a Stopping Time

False Alarm

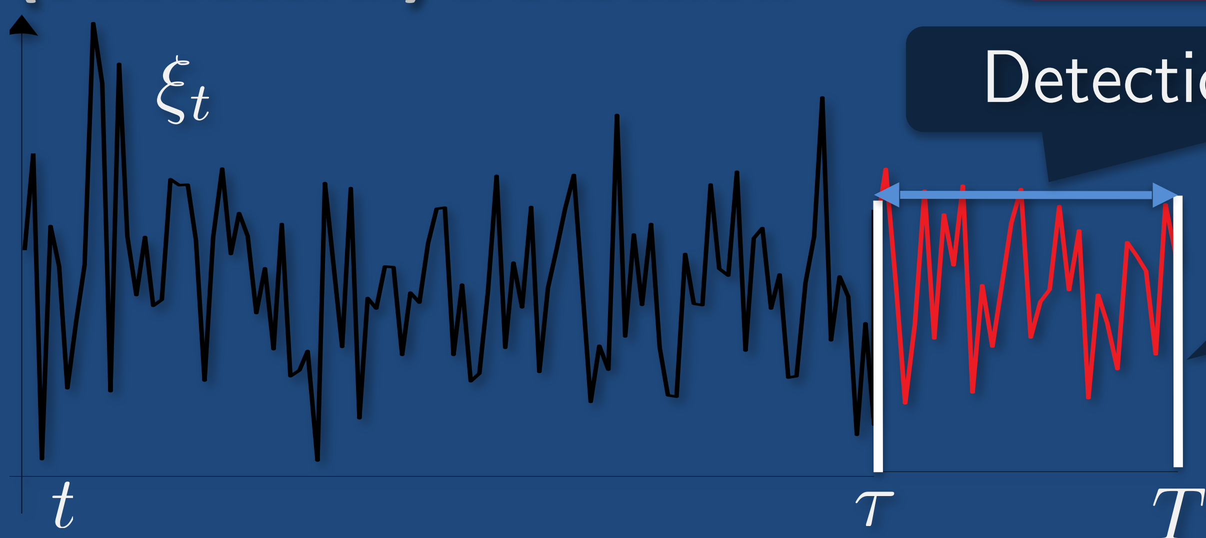


False Alarm Period

Probability of False Alarms

Design T to:
Minimize Detection Delay
or Maximize Detection
Probability, while controlling
False Alarms

(Successful) Detection



Detection Delay $T - \tau$

Probability of Detection

Quality monitoring of manufacturing process

Production line → Continuous Measurements → Quality Assessment

Medical Applications

Epidemic Detection

Disease rate measurements → Increase in rate? → Epidemic outbreak?

Early Detection of Epilepsy Episode

EEG Wearables → Divergence from normal → Episode?

Financial Applications

- Structural Change-detection in Exchange Rates
- Portfolio Monitoring

Electronic Communications

Seismology

Speech & Image Processing (segmentation)

Vibration monitoring (Structural health monitoring)

Security monitoring (fraud detection)

Spectrum monitoring

Scene monitoring

Network monitoring (router failures, attack detection)

⋮

Model for Change Detection Mechanism

τ

Stopping time

Dependent $\{X_t\}$, $\{\xi_t\}$

X_t: Vibrations at points of the structure

$\xi_t = AX_t + W_t$: Low dimensional noisy observations

If $\|X_t\|^2 \geq \nu$, then crack (change in structure)

Mechanism consults $X_1, \ldots, X_t$ and decides to apply change (**stop** nominal behavior) or not

Independent $\{X_t\}$, $\{\xi_t\}$

X_t: Coordinates of ball

ξ_t: Noisy sensor observations

If X_t inside the volume under the goal, “apply” change

Performance Metrics

Detection Delay

$$\mathcal{J}(T) = \mathsf{E}_1[T - \tau | T > \tau]$$

Hard Limited Delay

$$\mathcal{P}(T) = \mathsf{P}_1(T \leq \tau + M | T > \tau)$$

If stopping rule used by change mechanism unknown computation of $\mathcal{J}(T), \mathcal{P}(T)$ is not possible

We follow a **worst-case analysis**

Detection Delay

$$\mathcal{J}(T) = \sup_{\tau} \mathsf{E}_1[T-\tau|T>\tau]$$

$$= \sup_{t>0} \sup_{\xi_1,\ldots,\xi_t} \mathsf{E}_1[T-t|T>t,\xi_1,\ldots,\xi_t]$$

Lorden (1971)

$$= \sup_{t>0} \mathsf{E}_1[T-t|T>t] \text{ if } \{X_t\},\{\xi_t\} \text{ independent}$$

Pollak (1985)

Hard Limited Delay

$$\mathcal{P}(T) = \inf_{\tau} \mathsf{P}_1(T\le\tau+M|T>\tau)$$

$$= \inf_{t>0} \inf_{\xi_1,\ldots,\xi_t} \mathsf{P}_1(T\le t+M|T>t,\xi_1,\ldots,\xi_t)$$

$$= \inf_{t>0} \mathsf{P}_1(T\le t+M|T>t) \text{ if } \{X_t\},\{\xi_t\} \text{ independent}$$

Detection Delay

$$\inf_T \mathcal{J}(T) =$$

$$\inf_T \sup_{t>0} \sup_{\xi_1,\ldots,\xi_t} \mathsf{E}_1[T-t|T>t,\xi_1,\ldots,\xi_t]$$

$$\text{subject to}: \quad \mathsf{E}_0[T] \geq \gamma > 1$$

Hard Limited Delay

$$\sup_T \mathcal{P}(T) =$$

$$\sup_T \inf_{t>0} \inf_{\xi_1,\ldots,\xi_t} \mathsf{P}_1(T \leq t+M|T>t,\xi_1,\ldots,\xi_t)$$

$$\text{subject to}: \quad \mathsf{E}_0[T] \geq \gamma > 1$$

Optimum Detectors

Shiryaev 1963

$\{\xi_t\}$ are i.i.d. before and after the change with corresponding pdfs $\mathrm{f}_0(\xi)$, $\mathrm{f}_1(\xi)$

τ is independent from $\{\xi_t\}$ and follows a geometric distribution $\mathrm{P}(\tau = t) = p(1-p)^t, t = 0, 1, \ldots$

$$S_t = (1 + S_{t-1})\frac{\mathrm{f}_1(\xi_t)}{(1-p)\mathrm{f}_0(\xi_t)}$$

$$T_{\mathrm{S}} = \inf_{t>0}\{S_t \geq \nu\}$$

Threshold to satisfy false alarm constraint with equality

Lorden 1971

$\{\xi_t\}$ are i.i.d. before and after the change with corresponding pdfs $\mathsf{f}_0(\xi)$, $\mathsf{f}_1(\xi)$

τ and $\{\xi_t\}$ are dependent

$$\inf_T \mathcal{J}(T) = \inf_T \sup_{t>0} \sup_{\xi_1,\ldots,\xi_t} \mathsf{E}_1[T-t|T>t,\xi_1,\ldots,\xi_t]$$

$$\text{subject to : } \mathsf{E}_0[T] \geq \gamma > 1$$

$$S_t = \max\{S_{t-1},0\} + \log\frac{\mathsf{f}_1(\xi_t)}{\mathsf{f}_0(\xi_t)}$$

$$T_{\mathrm{C}} = \inf_{t>0}\{S_t \geq \nu\}$$

Threshold to satisfy false alarm constraint with equality

CUSUM test

CUSUM test

$$S_t = \max_{t>\tau\geq 0} \sum_{k=\tau+1}^{t} \log\frac{\mathrm{f}_1(\xi_k)}{\mathrm{f}_0(\xi_k)} \qquad S_t = \max\{S_{t-1}, 0\} + \log\frac{\mathrm{f}_1(\xi_t)}{\mathrm{f}_0(\xi_t)}$$

Known since 1954 as the **Page test**

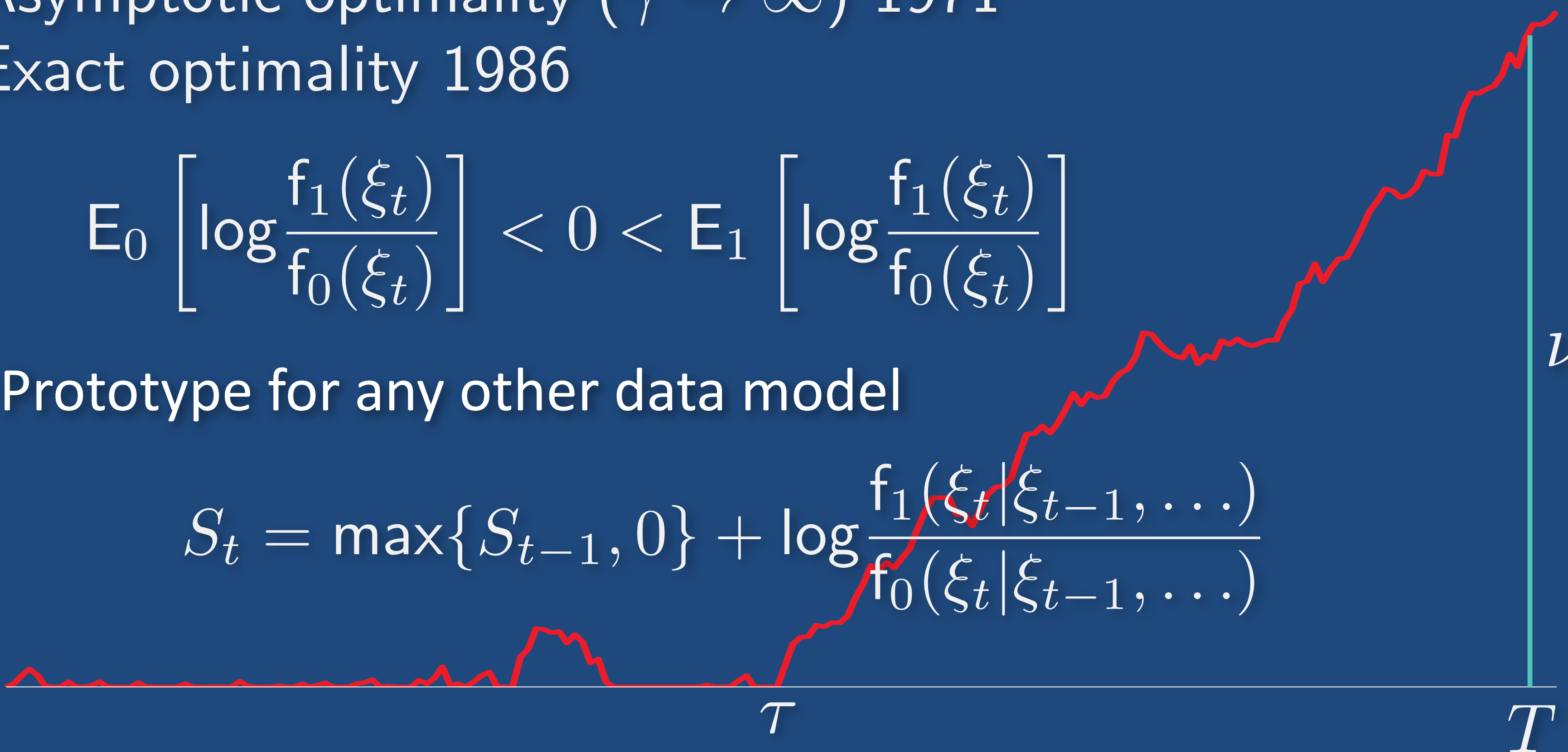
Asymptotic optimality $(\gamma \to \infty)$ 1971

Exact optimality 1986

$$\mathrm{E}_0\left[\log\frac{\mathrm{f}_1(\xi_t)}{\mathrm{f}_0(\xi_t)}\right] < 0 < \mathrm{E}_1\left[\log\frac{\mathrm{f}_1(\xi_t)}{\mathrm{f}_0(\xi_t)}\right]$$

Prototype for any other data model

$$S_t = \max\{S_{t-1}, 0\} + \log\frac{\mathrm{f}_1(\xi_t|\xi_{t-1},\ldots)}{\mathrm{f}_0(\xi_t|\xi_{t-1},\ldots)}$$



$\{\xi_t\}$ Markov before and after the change with corresponding pdfs $\mathsf{f}_0(\xi_t|\xi_{t-1})$, $\mathsf{f}_1(\xi_t|\xi_{t-1})$

Conjecture

$$S_t = \max\{S_{t-1}, \phi(\xi_{t-1})\} + \log \frac{\mathsf{f}_1(\xi_t|\xi_{t-1})}{\mathsf{f}_0(\xi_t|\xi_{t-1})}$$

$$T_C = \inf_t \{S_t \geq \nu(\xi_t)\}$$

Functions $\phi(\xi), \nu(\xi)$ are solution to a system of integral equations. Computed either numerically or asymptotically $\gamma \to \infty$

As $\gamma \to \infty$ we have $\phi(\xi) \to 0$ and $\nu(\xi) \to \nu$

Pollak 1985

$\{\xi_t\}$ are i.i.d. before and after the change with corresponding pdfs $\mathsf{f}_0(\xi)$, $\mathsf{f}_1(\xi)$

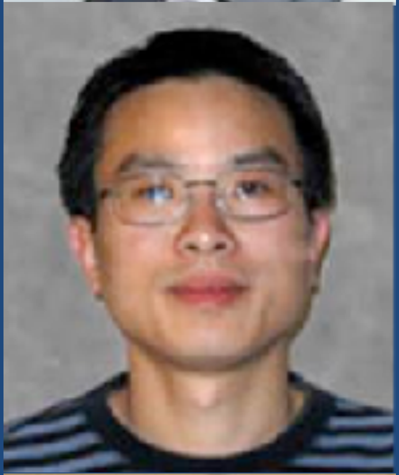
τ and $\{\xi_t\}$ are independent

$$\inf_T \mathcal{J}(T) = \inf_T \sup_{t>0} \mathsf{E}_1[T-t|T>t]$$

$$\text{subject to}: \quad \mathsf{E}_0[T] \geq \gamma > 1$$

$$S_t = (S_{t-1}+1)\frac{\mathsf{f}_1(\xi_t)}{\mathsf{f}_0(\xi_t)} \qquad T_{\mathrm{SRP}} = \inf_t \{S_t \geq \nu\}$$

Shiryaev-Roberts-Pollak test



Mei 2006

If S_0 specially designed T_{SRP} asymptotically ($\gamma \to \infty$) optimum.

Tartakovsky 2010

Exact optimality? 1997- 2006

Tartakovsky 2019

τ and $\{\xi_t\}$ are independent

$$\inf_T \mathcal{J}(T) = \inf_T \sup_{t>0} \mathsf{E}_1[T - t | T > t]$$

$$\text{subject to}: \quad \mathsf{E}_0[T] \geq \gamma > 1$$

Multiple post-change possibilities. Data after change i.i.d. with pdf $\mathsf{f}_1(\xi), \ldots, \mathsf{f}_k(\xi)$

$$S_t^i = (S_{t-1}^i + 1)\frac{\mathsf{f}_i(\xi_t)}{\mathsf{f}_0(\xi_t)}, \; i = 1, \ldots, k$$

$$T_{\mathrm{T}} = \inf_{t>0}\{S_t^1 + \cdots + S_t^k \geq \nu\}$$

T_{T} asymptotically $(\gamma \to \infty)$ optimum

Hard Limited Delay

τ and $\{\xi_t\}$ are dependent

$$\sup_T \mathcal{P}(T) = \sup_T \inf_{t>0} \inf_{\xi_1,\ldots,\xi_t} \mathsf{P}_1(T \le t+M | T > t, \xi_1, \ldots, \xi_t)$$

$$\text{subject to}: \quad \mathsf{E}_0[T] \ge \gamma > 1$$

τ and $\{\xi_t\}$ are independent

$$\sup_T \mathcal{P}(T) = \sup_T \inf_{t>0} \mathsf{P}_1(T \le t+M | T > t)$$

$$\text{subject to}: \quad \mathsf{E}_0[T] \ge \gamma > 1$$

If $\{\xi_t\}$ are i.i.d. before and after the change with pdfs $\mathsf{f}_0(\xi)$, $\mathsf{f}_1(\xi)$, and interested in $M = 1$ (detect immediately)

$$T_{\mathrm{Sh}} = \inf_{t>0} \left\{ \frac{\mathsf{f}_1(\xi_t)}{\mathsf{f}_0(\xi_t)} \ge \nu \right\}$$

Shewhart test (1931)

Markovian pre- and post-change pdfs for observations

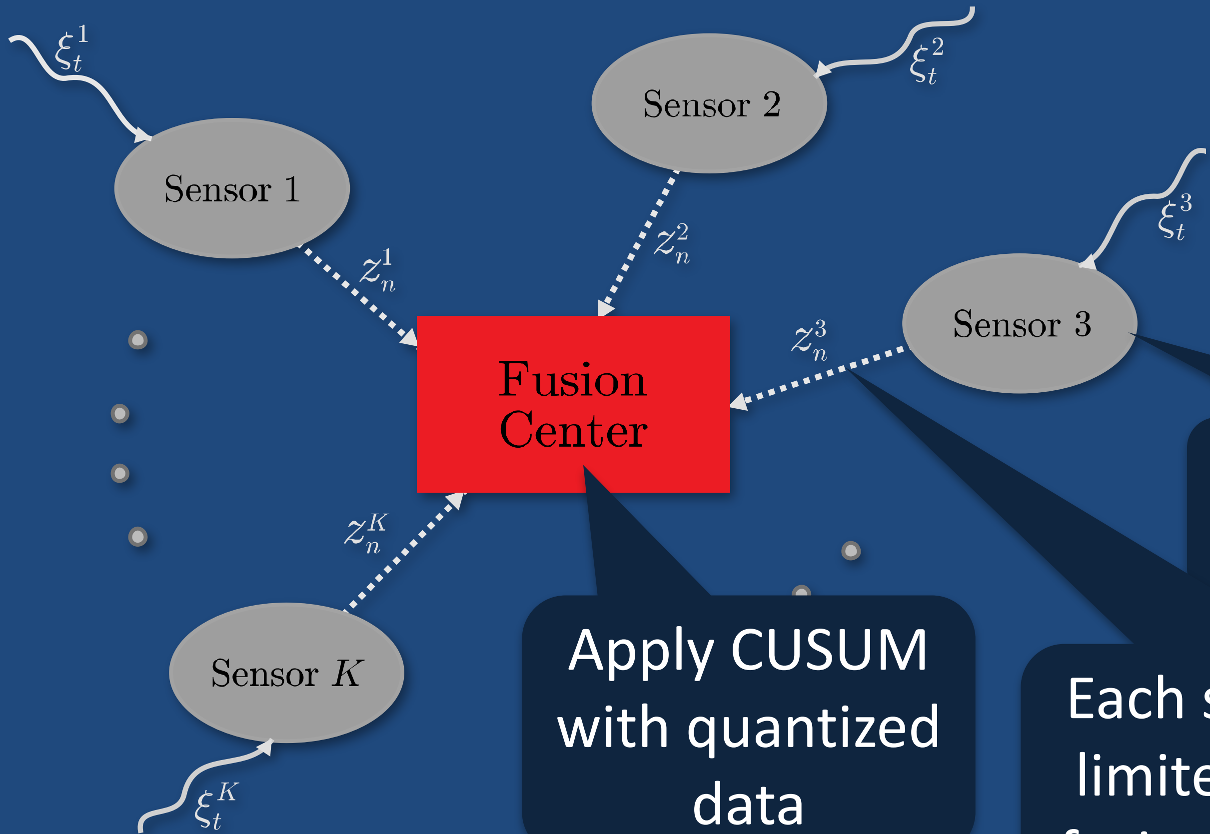
$$\sup_T \mathcal{P}(T) = \sup_T \inf_{t>0} \inf_{\xi_1,\ldots,\xi_t} \mathsf{P}_1(T = t+1 | T > t, \xi_1, \ldots, \xi_t)$$

$$\text{subject to}: \quad \mathsf{E}_0[T] \geq \gamma > 1$$

$$T_{\mathrm{Sh}} = \inf_{t>0}\left\{ c(\xi_{t-1}) \frac{\mathsf{f}_1(\xi_t | \xi_{t-1})}{\mathsf{f}_0(\xi_t | \xi_{t-1})} \geq \nu(\xi_t) \right\}$$

2015: Functions $c(\xi), \nu(\xi)$ satisfy system of integral equation. Can be computed numerically or asymptotically $(\gamma \to \infty)$. Simple solution for conditionally Gaussian: $\xi_t = \alpha(\xi_{t-1}) + w_t$ where $\{w_t\}$ i.i.d. $\mathcal{N}(0,1)$

Veeravalli 2001



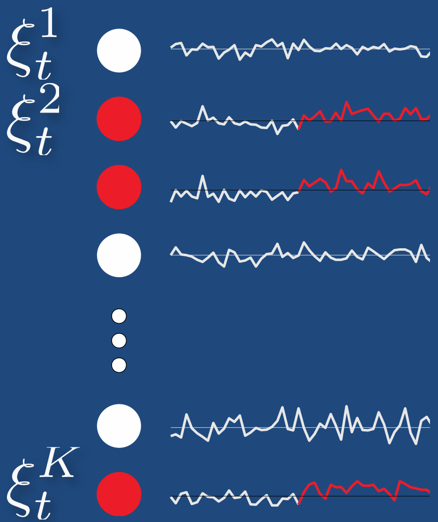
Simultaneous change in all sensors

Apply CUSUM with quantized data

Each sensor must send limited information to fusion center (i.e. 1 bit)

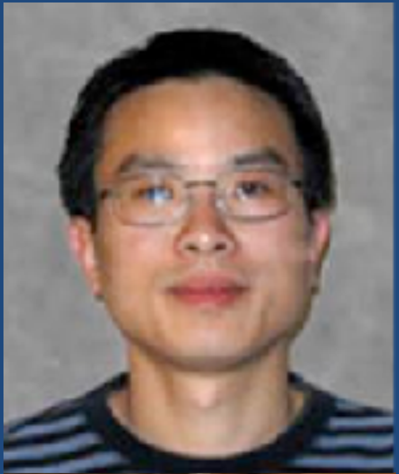
Apply optimum quantization on each likelihood ratio $\frac{f_1^i(\xi_t)}{f_0^i(\xi_t)}$

Mei 2011 - Fellouris 2016



There is a change in statistical behavior in an unknown subset.

ONLY DETECT



$$S_t^i = \max\{S_{t-1}^i, 0\} + \log \frac{f_1^i(\xi_t^i)}{f_0^i(\xi_t^i)}$$

$$T = \inf_{t>0} \{\max_i S_t^i \geq \nu\}$$

$$T = \inf_{t>0} \{S_t^1 + \cdots + S_t^K \geq \nu\}$$

$$T = \inf_{t>0} \{ \sum_{i=1}^K \log\left(1 - \pi + \pi e^{S_t^i}\right) \geq \nu\}$$

Unknown Statistics

What if $\mathsf{f}_1(\xi)$ and/or $\mathsf{f}_0(\xi)$ are not known

Lai 1998

$\mathsf{f}_0(\xi)$: known exactly (estimated from historical data)

$\mathsf{f}_1(\xi,\theta)$: where θ unknown parameters

$$S_t = \max_{t>\tau\geq 0} \sum_{k=\tau+1}^{t} \log\frac{\mathsf{f}_1(\xi_k,\theta)}{\mathsf{f}_0(\xi_k)}$$

$$T_{\mathrm{C}} = \inf_{t>0}\{S_t \geq \nu\}$$

$$S_t = \max_{t>\tau\geq 0} \sup_{\theta} \sum_{k=\tau+1}^{t} \log\frac{\mathsf{f}_1(\xi_k,\theta)}{\mathsf{f}_0(\xi_k)}$$

Asymptotic Optimality

$$S_t = \max_{t>\tau\geq t-w} \sup_{\theta} \sum_{k=\tau+1}^{t} \log\frac{\mathsf{f}_1(\xi_k,\theta)}{\mathsf{f}_0(\xi_k)}$$

Xie 2022

$\mathrm{f}_0(\xi)$: known exactly

$\mathrm{f}_1(\xi, \theta)$: where θ unknown parameters

$$S_t = \max\{S_{t-1}, 0\} + \log \frac{\mathrm{f}_1(\xi_t, \theta)}{\mathrm{f}_0(\xi_t)} \qquad T_{\mathrm{C}} = \inf_{t>0}\{S_t \geq \nu\}$$

$$S_t = \max\{S_{t-1}, 0\} + \log \frac{\mathrm{f}_1(\xi_t, \hat{\theta}_{t-1})}{\mathrm{f}_0(\xi_t)}$$

$\hat{\theta}_t$: Estimate of θ using $\{\xi_t, \ldots, \xi_{t-w+1}\}$

Asymptotic Optimality

Veeravalli 2022

$f_0(\xi)$: known exactly

$f_1(\xi)$: completely unknown

$$S_t = \max\{S_{t-1}, 0\} + \log\frac{f_1(\xi_t)}{f_0(\xi_t)}$$

$$T_C = \inf_{t>0}\{S_t \geq \nu\}$$

$$S_t = \max\{S_{t-1}, 0\} + \log\frac{\hat{f}_{1,t-1}(\xi_t)}{f_0(\xi_t)}$$

$\hat{f}_{1,t}(\xi)$: density estimate based on $\{\xi_t, \ldots, \xi_{t-w+1}\}$ using for example the Kernel method

Asymptotic Optimality

$\mathsf{f}_0(\xi), \mathsf{f}_1(\xi)$: completely unknown

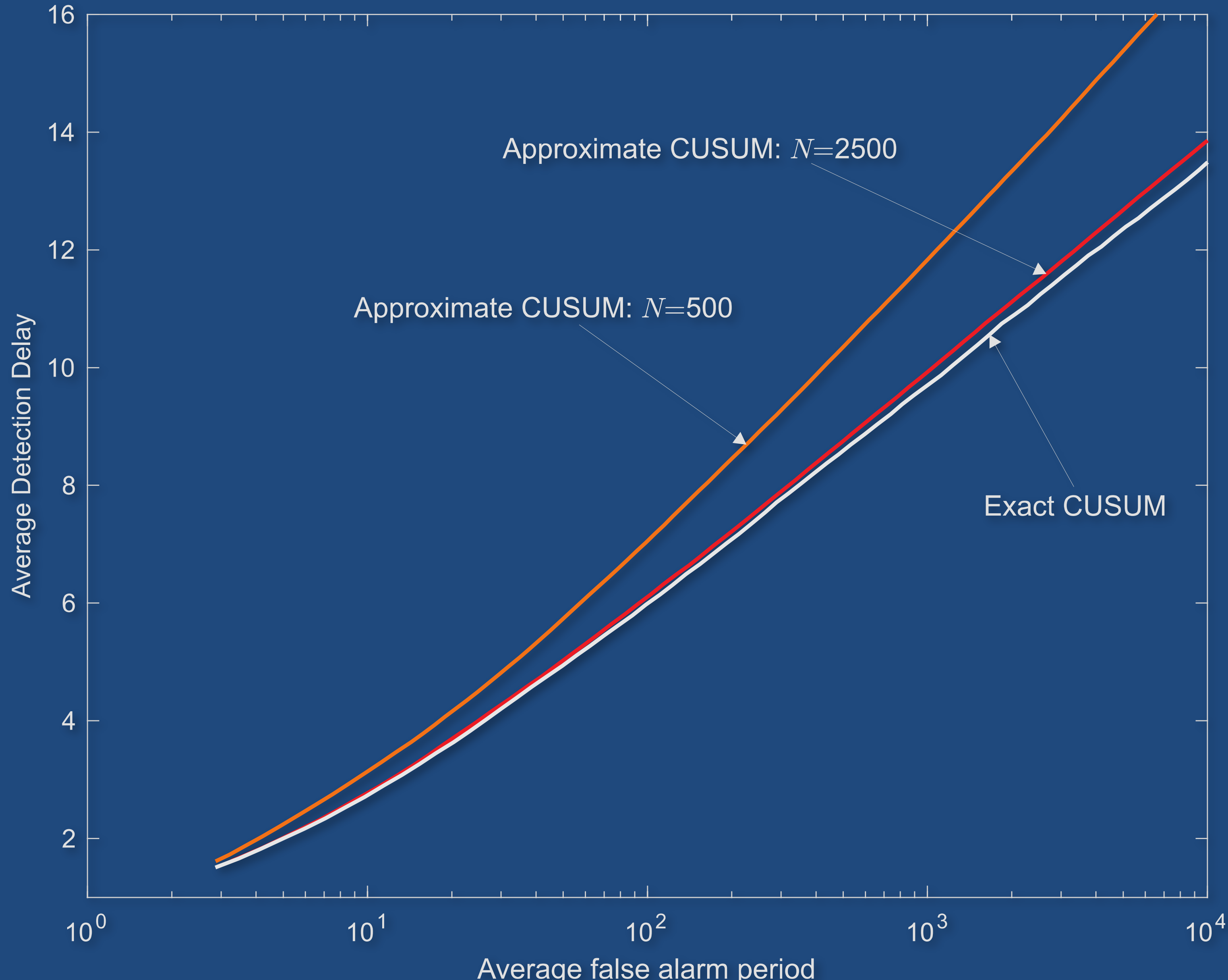
$\{\xi_1^0, \ldots, \xi_N^0\}$ sampled from pre-change $\mathsf{f}_0(\xi)$
$\{\xi_1^1, \ldots, \xi_N^1\}$ sampled from pre-change $\mathsf{f}_1(\xi)$

$$S_t = \max\{S_{t-1}, 0\} + \log\frac{\mathsf{f}_1(\xi_t)}{\mathsf{f}_0(\xi_t)}$$

Instead of estimating separately $\mathsf{f}_0(\xi), \mathsf{f}_1(\xi)$ we can estimate (with neural network) the log-likelihood ratio

$$u(\xi) = \log\frac{\mathsf{f}_1(\xi)}{\mathsf{f}_0(\xi)}$$

using the two data sets



Approximate CUSUM: N=2500
Approximate CUSUM: N=500
Exact CUSUM
Average Detection Delay
Average false alarm period
16
14
12
10
8
6
4
2
10^0
10^1
10^2
10^3
10^4