Data Driven Detection Methods

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Outline

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Hypothesis Testing

Mathematical Formulation

For a random vector X we assume the following two hypotheses

$$\begin{aligned} \mathsf{H}_0: &\quad X \sim \mathsf{f}_0(X),\ \ \mathbb{P}(\mathsf{H}_0) \\ \mathsf{H}_1: &\quad X \sim \mathsf{f}_1(X),\ \ \mathbb{P}(\mathsf{H}_1) \end{aligned}$$

For every X need to decide if it comes from H_0 or H_1

Decide using a *Decision Function* $\mathsf{D}(X) \in \{0, 1\}$

Would like to optimize $\mathsf{D}(X)$

Plethora of applications in diverse scientific fields!!!

Bayesian Approach

Minimize decision error probability

$$\min_{\mathsf{D}}\Big\{\mathbb{P}(\mathsf{D}=1|\mathsf{H}_0)\mathbb{P}(\mathsf{H}_0)+\mathbb{P}(\mathsf{D}=0|\mathsf{H}_1)\mathbb{P}(\mathsf{H}_1)\Big\}$$

$$\frac{\mathsf{f}_1(X)}{\mathsf{f}_0(X)} \underset{\mathsf{H}_0}{\overset{\mathsf{H}_1}{\gtreqless}} \frac{\mathbb{P}(\mathsf{H}_0)}{\mathbb{P}(\mathsf{H}_1)} \quad \equiv \quad \frac{\mathsf{f}_1(X)\mathbb{P}(\mathsf{H}_1)}{\mathsf{f}_0(X)\mathbb{P}(\mathsf{H}_0)} \underset{\mathsf{H}_0}{\overset{\mathsf{H}_1}{\gtreqless}} 1$$

For $\omega(\mathsf{r})$ strictly increasing

$$\mathsf{r}(X) \underset{\mathsf{H}_0}{\overset{\mathsf{H}_1}{\gtreqless}} 1 \;\equiv\; \omega\big(\mathsf{r}(X)\big) \underset{\mathsf{H}_0}{\overset{\mathsf{H}_1}{\gtreqless}} \omega(1), \qquad \mathsf{r}(X)=\frac{\mathsf{f}_1(X)\mathbb{P}(\mathsf{H}_1)}{\mathsf{f}_0(X)\mathbb{P}(\mathsf{H}_0)}$$

Neyman-Pearson Approach

$$\mathsf{H}_0: \quad X \sim \mathsf{f}_0(X), \quad \mathbb{P}(\mathsf{H}_0)$$
$$\mathsf{H}_1: \quad X \sim \mathsf{f}_1(X), \quad \mathbb{P}(\mathsf{H}_1)$$

Maximize detection probability $\mathbb{P}(\mathsf{D}=1|\mathsf{H}_1)$

subject to false alarm probability constraint $\mathbb{P}(\mathsf{D}=1|\mathsf{H}_0) \le \alpha$

$$\frac{\mathsf{f}_1(X)}{\mathsf{f}_0(X)} \underset{\mathsf{H}_0}{\overset{\mathsf{H}_1}{\gtreqless}} \lambda, \quad \mathbb{P}\left(\frac{\mathsf{f}_1(X)}{\mathsf{f}_0(X)} \ge \lambda \Big| \mathsf{H}_0\right) = \alpha$$

For $\omega(\mathsf{r})$ strictly increasing

$$\omega\big(\mathsf{r}(X)\big) \underset{\mathsf{H}_0}{\overset{\mathsf{H}_1}{\gtreqless}} \eta, \quad \mathbb{P}\Big(\omega\big(\mathsf{r}(X)\big) \ge \eta \Big| \mathsf{H}_0\Big) = \alpha, \qquad \mathsf{r}(X) = \frac{\mathsf{f}_1(X)}{\mathsf{f}_0(X)}$$

Data Driven Approach

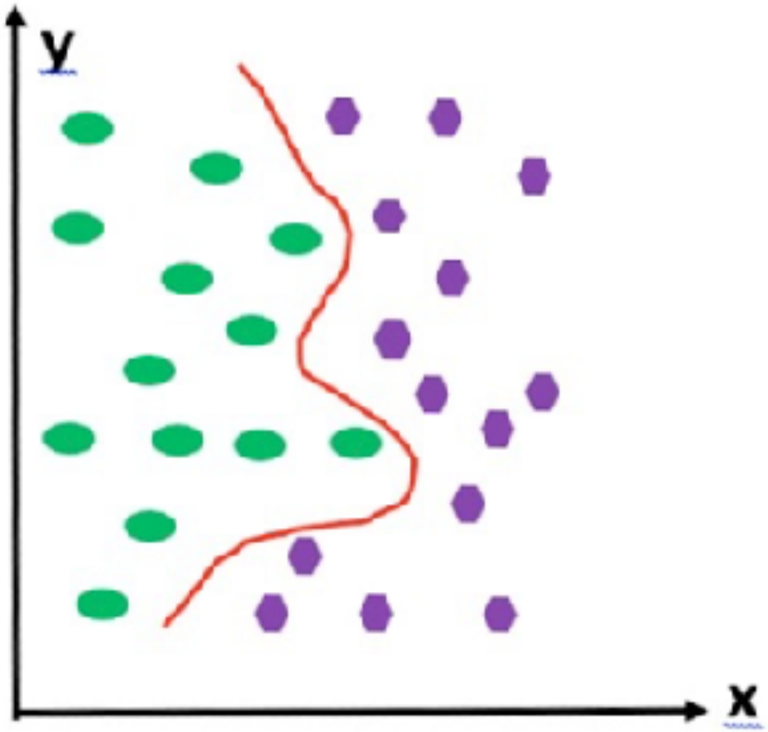
$\mathsf{H}_0: \quad X \sim \mathsf{f}_0(X), \quad \mathbb{P}(\mathsf{H}_0)$

$\mathsf{H}_1: \quad X \sim \mathsf{f}_1(X), \quad \mathbb{P}(\mathsf{H}_1)$

$X_1^0 \; X_2^0 \; \ldots \; X_{n_0}^0$ — Sampled from f_0

$X_1^1 \; X_2^1 \; \ldots \; X_{n_1}^1$ — Sampled from f_1

$$\mathbb{P}(\mathsf{H}_i) \approx \frac{n_i}{n_0 + n_1}$$



Design border to separate the two datasets

What is the best border ?

$$\text{All } X: \; \frac{\mathsf{f}_1(X)\mathbb{P}(\mathsf{H}_1)}{\mathsf{f}_0(X)\mathbb{P}(\mathsf{H}_0)} = 1$$

Instead of a "border", design a decision like function $\mathsf{v}(X)$

$$\mathsf{v}(X) = \begin{cases} -1 & \text{when } X \text{ from } \mathsf{H}_0 \\ 1 & \text{when } X \text{ from } \mathsf{H}_1. \end{cases}$$

Use parametric family of functions $\mathsf{u}(X,\theta)$ and optimize θ solving

$$\mathsf{J}(\theta) = \frac{1}{n_0 + n_1} \left\{ \sum_{i=1}^{n_0} \Big(-1 - \mathsf{u}(X_i^0, \theta) \Big)^2 + \sum_{j=1}^{n_1} \Big(1 - \mathsf{u}(X_j^1, \theta) \Big)^2 \right\}$$

$$\min_{\theta} \mathsf{J}(\theta) \;\Rightarrow\; \theta_{\mathsf{o}} \;\Rightarrow\; \mathsf{u}(X, \theta_{\mathsf{o}})$$

For every X to test decide as follows: $\mathsf{u}(X, \theta_{\mathsf{o}}) \overset{\mathsf{H}_1}{\underset{\mathsf{H}_0}{\gtreqless}} 0$

Works “well”!! Why??

Understanding using Asymptotic Analysis

$$n_0, n_1 \to \infty, \qquad \mathsf{u}(X,\theta) \to \mathsf{v}(X)$$

$$\mathsf{J}(\theta) = \frac{n_0}{n_0+n_1} \frac{1}{n_0} \sum_{i=1}^{n_0} \Big(1 + \mathsf{u}(X_i^0,\theta)\Big)^2 + \frac{n_1}{n_0+n_1} \frac{1}{n_1} \sum_{j=1}^{n_1} \Big(1 - \mathsf{u}(X_j^1,\theta)\Big)^2$$

$$\mathsf{J}(\mathsf{v}) = \mathbb{P}(\mathsf{H}_0)\mathbb{E}_0\left[\Big(1+\mathsf{v}(X)\Big)^2\right] + \mathbb{P}(\mathsf{H}_1)\mathbb{E}_1\left[\Big(1-\mathsf{v}(X)\Big)^2\right]$$

$$\min_\theta \mathsf{J}(\theta) \to \min_\mathsf{v} \mathsf{J}(\mathsf{v})$$

$$\theta_o \Rightarrow \mathsf{u}(X,\theta_o) \approx \mathsf{v}_o(X)$$

$$\mathbb{E}_1\left[\Big(1-\mathsf{v}(X)\Big)^2\right]=\mathbb{E}_0\left[\Big(1-\mathsf{v}(X)\Big)^2\frac{\mathsf{f}_1(X)}{\mathsf{f}_0(X)}\right]$$

$$\mathsf{J}(\mathsf{v})=\mathbb{P}(\mathsf{H}_0)\mathbb{E}_0\left[\Big(1+\mathsf{v}(X)\Big)^2+\mathsf{r}(X)\Big(1-\mathsf{v}(X)\Big)^2\right] \qquad \mathsf{r}(X)=\frac{\mathsf{f}_1(X)\mathbb{P}(\mathsf{H}_1)}{\mathsf{f}_0(X)\mathbb{P}(\mathsf{H}_0)}$$

minimize for each X

$$\mathsf{v}_o(X)=\frac{\mathsf{r}(X)-1}{\mathsf{r}(X)+1}=\omega\big(\mathsf{r}(X)\big), \quad \text{where } \omega(\mathsf{r})=\frac{\mathsf{r}-1}{\mathsf{r}+1} \text{ strictly increasing}$$

Test equivalent to Bayes: $\mathsf{v}_o(X)=\omega\big(\mathsf{r}(X)\big)\underset{\mathsf{H}_0}{\overset{\mathsf{H}_1}{\gtreqless}}\omega(1)=0 \quad\Rightarrow\quad \mathsf{u}(X,\theta_o)\underset{\mathsf{H}_0}{\overset{\mathsf{H}_1}{\gtreqless}}0$

Equivalence in the limit

Consistency (with respect to the Bayes test)

$$\min_{\theta} \mathsf{J}(\theta) = \min_{\theta} \left\{ \sum_{i=1}^{n_0} \Big(1 + \mathsf{u}(X_i^0, \theta)\Big)^2 + \sum_{j=1}^{n_1} \Big(1 - \mathsf{u}(X_j^1, \theta)\Big)^2 \right\}$$

$$\Rightarrow \ \theta_{\mathsf{o}} \ \Rightarrow \ \mathsf{u}(X, \theta_{\mathsf{o}})$$

$$\min_{\mathsf{v}} \mathsf{J}(\mathsf{v}) = \min_{\mathsf{v}} \left\{ \mathbb{P}(\mathsf{H}_0)\mathbb{E}_0 \left[\Big(1 + \mathsf{v}(X)\Big)^2 \right] + \mathbb{P}(\mathsf{H}_1)\mathbb{E}_1 \left[\Big(1 - \mathsf{v}(X)\Big)^2 \right] \right\}$$

$$\Rightarrow \ \mathsf{v}_{\mathsf{o}}(X) \ = \ \omega\big(\mathsf{r}(X)\big)$$

Expect: $\mathsf{u}(X, \theta_{\mathsf{o}}) \approx \omega\big(\mathsf{r}(X)\big)$

Optimum Test: $\omega\big(\mathsf{r}(X)\big) \overset{\mathsf{H}_1}{\underset{\mathsf{H}_0}{\gtreqless}} \omega(1)$, Close to Optimum: $\mathsf{u}(X, \theta_{\mathsf{o}}) \overset{\mathsf{H}_1}{\underset{\mathsf{H}_0}{\gtreqless}} \omega(1)$

Develop data driven methods for estimation of $\omega\big(\mathsf{r}(X)\big)$ for other $\omega(\mathsf{r})$

Consistent tests eventually prevail over inconsistent tests

Likelihood Ratio Estimation

$$\mathsf{r}(X) \underset{\mathsf{H}_0}{\overset{\mathsf{H}_1}{\gtreqless}} 1 \;\equiv\; \omega_1\big(\mathsf{r}(X)\big) \underset{\mathsf{H}_0}{\overset{\mathsf{H}_1}{\gtreqless}} \omega_1(1) \;\equiv\; \omega_2\big(\mathsf{r}(X)\big) \underset{\mathsf{H}_0}{\overset{\mathsf{H}_1}{\gtreqless}} \omega_2(1)$$

$$\mathsf{u}(X,\theta_{\mathsf{o}}) \underset{\mathsf{H}_0}{\overset{\mathsf{H}_1}{\gtreqless}} 1 \;\not\equiv\; \mathsf{u}_1(X,\theta_1) \underset{\mathsf{H}_0}{\overset{\mathsf{H}_1}{\gtreqless}} \omega_1(1) \;\not\equiv\; \mathsf{u}_2(X,\theta_2) \underset{\mathsf{H}_0}{\overset{\mathsf{H}_1}{\gtreqless}} \omega_2(1)$$

For function $\omega(\mathsf{r})$ can we define cost

$$\mathsf{J}(\mathsf{v}) = \mathbb{P}(\mathsf{H}_0)\mathbb{E}_0\left[\phi\big(\mathsf{v}(X)\big)\right] + \mathbb{P}(\mathsf{H}_1)\mathbb{E}_1\left[\psi\big(\mathsf{v}(X)\big)\right]$$

so that $\min_{\mathsf{v}} \mathsf{J}(\mathsf{v}) \;\Rightarrow\; \mathsf{v}_{\mathsf{o}}(X) = \omega\big(\mathsf{r}(X)\big)$?

THEOREM: Select strictly increasing function $\omega(\mathsf{r})$ and strictly negative function $\rho(z)$. Define

$$\psi'(z) = \rho(z), \quad \phi'(z) = -\omega^{-1}(z)\rho(z)$$

then the solution of the optimization problem

$$\min_{\mathsf{v}} \mathsf{J}(\mathsf{v}) = \min_{\mathsf{v}} \Big\{ \mathbb{P}(\mathsf{H}_0)\mathbb{E}_0\big[\phi\big(\mathsf{v}(X)\big)\big] + \mathbb{P}(\mathsf{H}_1)\mathbb{E}_1\big[\psi\big(\mathsf{v}(X)\big)\big] \Big\}$$

satisfies $\mathsf{v}_o(X) = \arg\min_{\mathsf{v}} \mathsf{J}(\mathsf{v}) = \omega\big(\mathsf{r}(X)\big)$

Same optimal solution $\omega(\mathsf{r}(X))$ for all functions $\rho(z) < 0$

Examples of functions

A: $\omega(\mathsf{r}) = \mathsf{r} \in \mathbb{R}_+$ (likelihood ratio)

$$\rho(z) = -1, z \geq 0 \;\Rightarrow\; \phi(z) = \frac{z^2}{2}, \;\; \psi(z) = -z$$

Mean Square

B: $\omega(\mathsf{r}) = \log(\mathsf{r}) \in \mathbb{R}$ (log-likelihood ratio)

$$\rho(z) = -e^{-0.5z} \;\Rightarrow\; \phi(z) = 2e^{0.5z}, \;\; \psi(z) = 2e^{-0.5z}$$

Exponential

C: $\omega(\mathsf{r}) = \frac{\mathsf{r}}{\mathsf{r}+1} \in [0,1]$ (posterior probability)

$$\rho(z) = -\frac{1}{z}, z \in [0,1] \;\Rightarrow\; \phi(z) = -\log(1-z), \;\; \psi(z) = -\log(z)$$

Cross Entropy

Data Driven Implementation

$$\mathsf{J}(\mathsf{v}) = \mathbb{P}(\mathsf{H}_0)\mathbb{E}_0\left[\phi\big(\mathsf{v}(X)\big)\right] + \mathbb{P}(\mathsf{H}_1)\mathbb{E}_1\left[\psi\big(\mathsf{v}(X)\big)\right]$$

$$\mathsf{J}(\theta) = \frac{1}{n_0+n_1}\left\{\sum_{i=1}^{n_0}\phi\big(\mathsf{u}(X_i^0,\theta)\big) + \sum_{j=1}^{n_1}\psi\big(\mathsf{u}(X_j^1,\theta)\big)\right\}$$

$$\mathsf{u}(X,\theta_{\mathsf{o}}) \approx \omega\left(\frac{\mathsf{f}_1(X)\mathbb{P}(\mathsf{H}_1)}{\mathsf{f}_0(X)\mathbb{P}(\mathsf{H}_0)}\right)$$

$$\mathsf{J}(\theta) = \frac{1}{n_0}\sum_{i=1}^{n_0}\phi\big(\mathsf{u}(X_i^0,\theta)\big) + \frac{1}{n_1}\sum_{j=1}^{n_1}\psi\big(\mathsf{u}(X_j^1,\theta)\big)$$

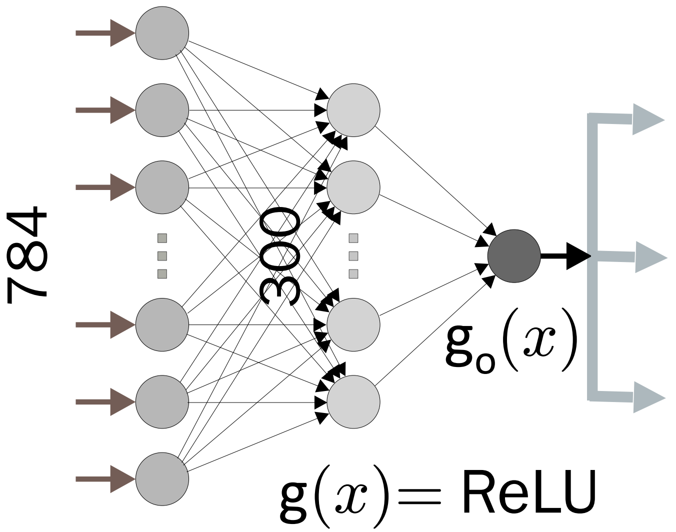
$$\mathsf{u}(X,\theta_{\mathsf{o}}) \approx \omega\left(\frac{\mathsf{f}_1(X)}{\mathsf{f}_0(X)}\right)$$

Example: Classification Problem

From dataset MNIST isolate handwritten numerals 4 and 9

Gray scale images 28 X 28 = 784 pixels. Design classifier using training data. Examine performance using testing data.

Neural network 784 X 300 X 1



Mean square $g_o(x)=$ ReLU, $u_1(X,\theta_1)$

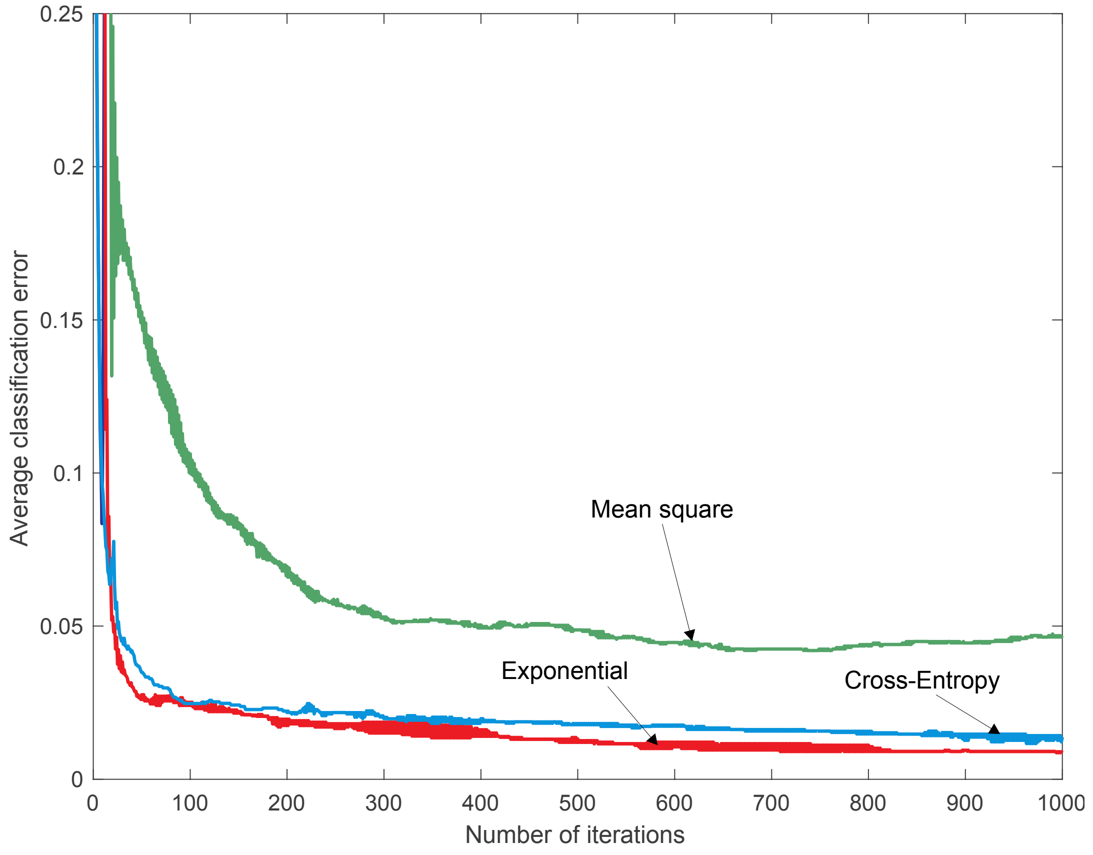
Exponential $g_o(x)= x$, $u_2(X,\theta_2)$

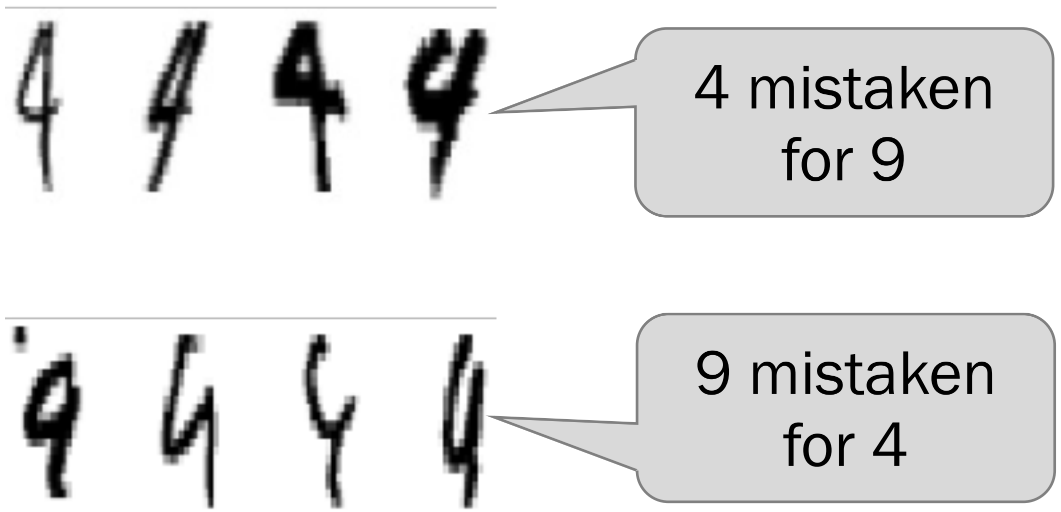
Cross entropy $g_o(x)=$ sigmoid, $u_3(X,\theta_3)$

# of parameters 235801

Training set: 5500 “4” and 5500 “9”. Testing set: 982 “4” and 1009 “9”

$$\mathsf{u}_1(X,\theta_1) \underset{\mathsf{H}_0}{\overset{\mathsf{H}_1}{\gtreqless}} 1, \quad \mathsf{u}_2(X,\theta_2) \underset{\mathsf{H}_0}{\overset{\mathsf{H}_1}{\gtreqless}} 0, \quad \mathsf{u}_3(X,\theta_3) \underset{\mathsf{H}_0}{\overset{\mathsf{H}_1}{\gtreqless}} \frac{1}{2}$$





Detection in Time Series

More practically interesting case: Testing of time series $\{ X_1 , X_2 ,...., X_n \}$

The whole set of measurements under H_0 or H_1

For testing we need likelihood ratio

$$\frac{\mathsf{f}_1(X_n,\ldots,X_1)}{\mathsf{f}_0(X_n,\ldots,X_1)} = \frac{\mathsf{f}_1(X_n|X_{n-1},\ldots,X_1)}{\mathsf{f}_0(X_n|X_{n-1},\ldots,X_1)} \frac{\mathsf{f}_1(X_{n-1}|X_{n-2},\ldots,X_1)}{\mathsf{f}_0(X_{n-1}|x_{X-2},\ldots,X_1)} \cdots \frac{\mathsf{f}_1(X_1)}{\mathsf{f}_0(X_1)}$$

When i.i.d. under each hypothesis

$$\frac{\mathsf{f}_1(X_n,\ldots,X_1)}{\mathsf{f}_0(X_n,\ldots,X_1)} = \frac{\mathsf{f}_1(X_n)}{\mathsf{f}_0(X_n)} \cdots \frac{\mathsf{f}_1(X_1)}{\mathsf{f}_0(X_1)}$$

Test to be used

$$\sum_{i=1}^{n} \log\left(\frac{\mathsf{f}_1(X_i)}{\mathsf{f}_0(X_i)}\right) \overset{\mathsf{H}_1}{\underset{\mathsf{H}_0}{\gtreqless}} \eta$$

Interested in estimating $\omega\big(\mathsf{r}(X)\big) = \omega\left(\frac{\mathsf{f}_1(X)}{\mathsf{f}_0(X)}\right)$

We are given training data:

$\{X_1^0, \ldots, X_{n_0}^0\}$ following H_0

$\{X_1^1, \ldots, X_{n_1}^1\}$ following H_1

For each $\omega(\mathsf{r})$ of interest, minimize corresponding $\mathsf{J}(\theta)$

$$\mathsf{J}(\theta) = \frac{1}{n_0}\sum_{i=1}^{n_0}\phi\big(\mathsf{u}(X_i^0,\theta)\big) + \frac{1}{n_1}\sum_{j=1}^{n_1}\psi\big(\mathsf{u}(X_j^1,\theta)\big)$$

$$\mathsf{u}(X,\theta_{\mathsf{o}}) \approx \omega\left(\frac{\mathsf{f}_1(X)}{\mathsf{f}_0(X)}\right)$$

$$\text{For } \omega(\mathsf{r}) = \mathsf{r}, \quad \mathsf{u}_1(X,\theta_1) \approx \frac{\mathsf{f}_1(X)}{\mathsf{f}_0(X)}, \quad \text{use} \quad \log\big(\mathsf{u}_1(X,\theta_1)\big) \quad \text{(Mean Square)}$$

$$\text{For } \omega(\mathsf{r}) = \log(\mathsf{r}), \quad \mathsf{u}_2(X,\theta_2) \approx \log\left(\frac{\mathsf{f}_1(X)}{\mathsf{f}_0(X)}\right), \quad \text{use} \quad \mathsf{u}_2(X,\theta_2) \quad \text{(Exponential)}$$

$$\text{For } \omega(\mathsf{r}) = \frac{\mathsf{r}}{\mathsf{r}+1}, \quad \mathsf{u}_3(X,\theta_3) \approx \frac{\frac{\mathsf{f}_1(X)}{\mathsf{f}_0(X)}}{\frac{\mathsf{f}_1(X)}{\mathsf{f}_0(X)}+1}, \quad \text{use} \quad \log\left(\frac{\mathsf{u}_3(X,\theta_3)}{1-\mathsf{u}_3(X,\theta_3)}\right)$$

(Cross Entropy)

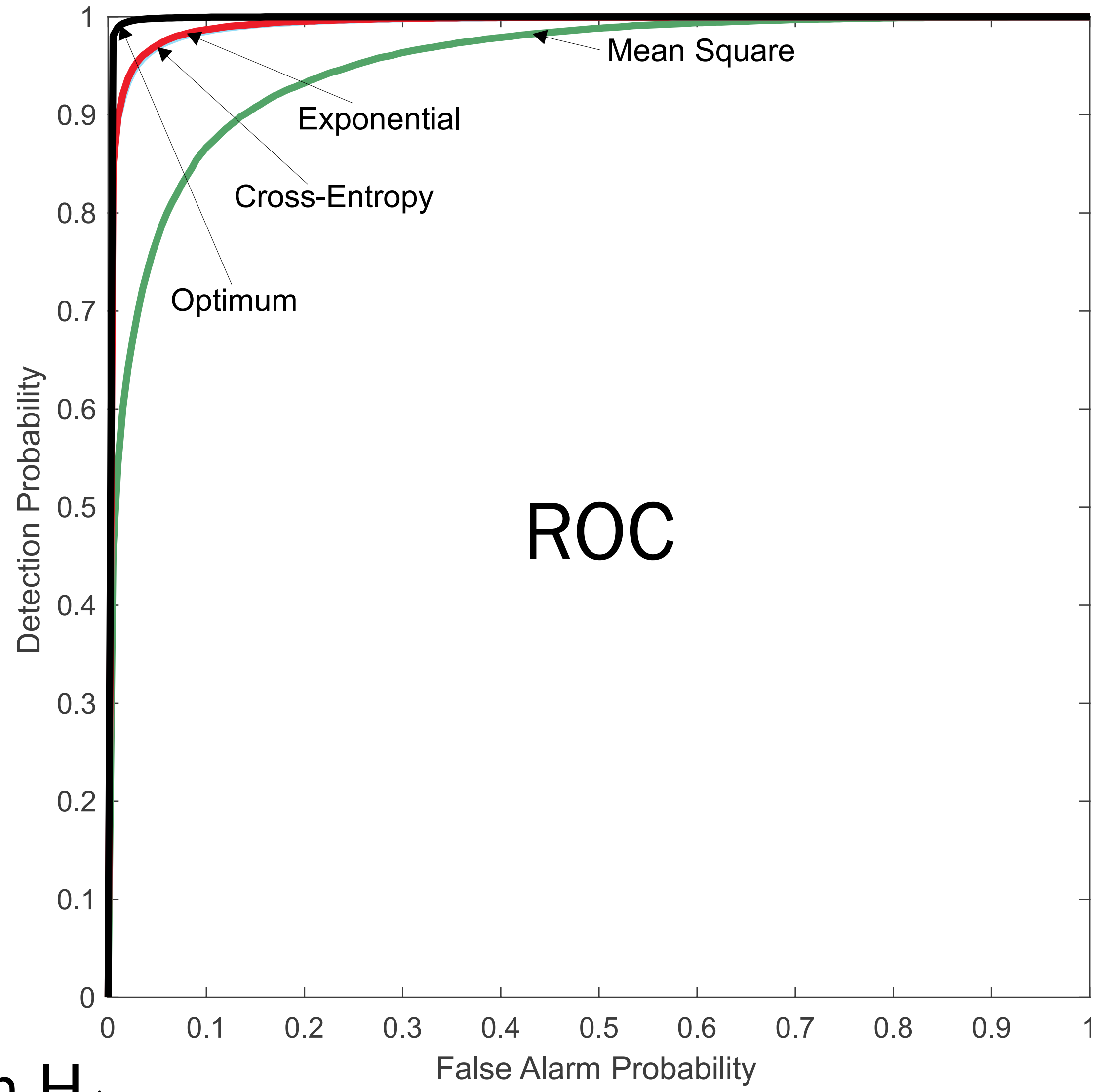
Example: Testing i.i.d. sequences

X_i length 10, $\mathsf{f}_0(X) \sim \mathcal{N}(0, I)$
$\mathsf{f}_1(X) \sim \mathcal{N}(\frac{1}{\sqrt{10}}[1 \cdots 1]^\intercal, 1.2I)$

Neural Network 10×20×1
Training data $n_0 = n_1 =$100
Produce $\mathsf{u}_1(X,\theta_1), \mathsf{u}_2(X,\theta_2), \mathsf{u}_3(X,\theta_3)$

Testing for $n =$20 samples

$$\left.\begin{array}{r} \sum_{i=1}^{20} \log\big(\mathsf{u}_1(X_i,\theta_1)\big) \\ \sum_{i=1}^{20} \mathsf{u}_2(X_i,\theta_2) \\ \sum_{i=1}^{20} \log\left(\frac{\mathsf{u}_3(X_i,\theta_3)}{1-\mathsf{u}_3(X_i,\theta_3)}\right) \end{array}\right\} \underset{\mathsf{H}_0}{\overset{\mathsf{H}_1}{\gtreqless}} \eta$$



100000×20 realizations from H_0 and from H_1

Markovian processes

Consider Markovian processes with "memory" m

$$\frac{\mathsf{f}_1(X_n|X_{n-1},\ldots,X_1)}{\mathsf{f}_0(X_n|X_{n-1},\ldots,X_1)}=\frac{\mathsf{f}_1(X_n|X_{n-1},\ldots,X_{n-m})}{\mathsf{f}_0(X_n|X_{n-1},\ldots,X_{n-m})}$$

$$\frac{\mathsf{f}_1(X_n,\ldots,X_1)}{\mathsf{f}_0(X_n,\ldots,X_1)}=\frac{\mathsf{f}_1(X_n|X_{n-1},\ldots,X_{n-m})}{\mathsf{f}_0(X_n|X_{n-1},\ldots,X_{n-m})}\cdots\frac{\mathsf{f}_1(X_{m+1}|X_m,\ldots,X_1)}{\mathsf{f}_0(X_{m+1}|X_m,\ldots,X_1)}\times\frac{\mathsf{f}_1(X_m,\ldots,X_1)}{\mathsf{f}_0(X_m,\ldots,X_1)}$$

Can we estimate likelihood ratio of conditional densities?

a) Through data dynamics (classical)

b) Directly (proposed)

Classical Approach

Most common model, Autoregressive

$$X_t = A_1^i X_{t-1} + \cdots + A_m^i X_{t-m} + W_t, \quad i = 0, 1$$

$$X_t = G_i(X_{t-1}, \ldots, X_{t-m}, \theta^i) + W_t$$

Use training data $\{X_1^0, \ldots, X_{n_0}^0\}$ and $\{X_1^1, \ldots, X_{n_1}^1\}$ to solve

$$\min_{\theta^i} \sum_{t=1}^{n_i} \left(X_t^i - G_i(X_{t-1}^i, \ldots, X_{t-m}^i, \theta^i) \right)^2 \quad \Rightarrow \quad \theta_\mathsf{o}^i$$

$$W_t^i = X_t^i - G_i(X_{t-1}^i, \ldots, X_{t-m}^i, \theta_\mathsf{o}^i), \quad \Sigma_i = \frac{1}{n_i} \sum_{t=1}^{n_i} W_t^i (W_t^i)^\intercal$$

Assume $\{W_t^i\}$ i.i.d. Gaussian $\mathcal{N}(0, \Sigma_i)$

X_t given $\{X_{t-1},\ldots,X_{t-m}\}$ under hypothesis H_i

Gaussian with mean $G_i(X_{t-1},\ldots,X_{t-m},\theta_\mathsf{o}^i)$ and covariance Σ_i

To test $\{X_1,\ldots,X_n\}$

$$W_t^i = X_t - G_i(X_{t-1},\ldots,X_{t-m},\theta_\mathsf{o}^i)$$

$$\frac{\mathsf{f}_1(X_t|X_{t-1},\ldots,X_{t-m})}{\mathsf{f}_0(X_t|X_{t-1},\ldots,X_{t-m})} = \frac{e^{-\frac{1}{2}(W_t^1)^\intercal\Sigma_1^{-1}W_t^1}}{e^{-\frac{1}{2}(W_t^0)^\intercal\Sigma_0^{-1}W_t^0}}\sqrt{\frac{|\Sigma_0|}{|\Sigma_1|}}$$

Not purely data driven

Gaussian assumption arbitrary, not necessarily suitable for all data!

Proposed Approach

$$\log\left(\frac{\mathsf{f}_1(X_t|X_{t-1},\ldots,X_{t-m})}{\mathsf{f}_0(X_t|X_{t-1},\ldots,X_{t-m})}\right) = \log\left(\frac{\frac{\mathsf{f}_1(X_t,X_{t-1},\ldots,X_{t-m})}{\mathsf{f}_1(X_{t-1},\ldots,X_{t-m})}}{\frac{\mathsf{f}_0(X_t,X_{t-1},\ldots,X_{t-m})}{\mathsf{f}_0(X_{t-1},\ldots,X_{t-m})}}\right) =$$

$$\log\left(\frac{\mathsf{f}_1(X_t,X_{t-1},\ldots,X_{t-m})}{\mathsf{f}_0(X_t,X_{t-1},\ldots,X_{t-m})}\right) - \log\left(\frac{\mathsf{f}_1(X_{t-1},\ldots,X_{t-m})}{\mathsf{f}_0(X_{t-1},\ldots,X_{t-m})}\right)$$

$$\mathsf{u}_{m+1}(X_t,\ldots,X_{t-m},\theta_{m+1}) \qquad \mathsf{u}_m(X_t,\ldots,X_{t-m+1},\theta_m)$$

$$\log\left(\frac{\mathsf{f}_1(X_t|X_{t-1},\ldots,X_{t-m})}{\mathsf{f}_0(X_t|X_{t-1},\ldots,X_{t-m})}\right) \approx$$

$$\mathsf{u}_{m+1}(X_t,\ldots,X_{t-m},\theta_{m+1}) - \mathsf{u}_m(X_{t-1},\ldots,X_{t-m},\theta_m)$$

Example: Testing Markov sequences (proof of concept)

Scalar observations $\{x_1, \ldots, x_n\}$

$w_t \sim \mathcal{N}(0,1)$, i.i.d.

$\mathsf{H}_0 : x_t = w_t$

$\mathsf{H}_1 : x_t = \mathsf{sign}(x_{t-1})\sqrt{|x_{t-1}|} + w_t$

$\mathsf{u}_2(x_t, x_{t-1}, \theta_2) : \ 2 \times 20 \times 1$

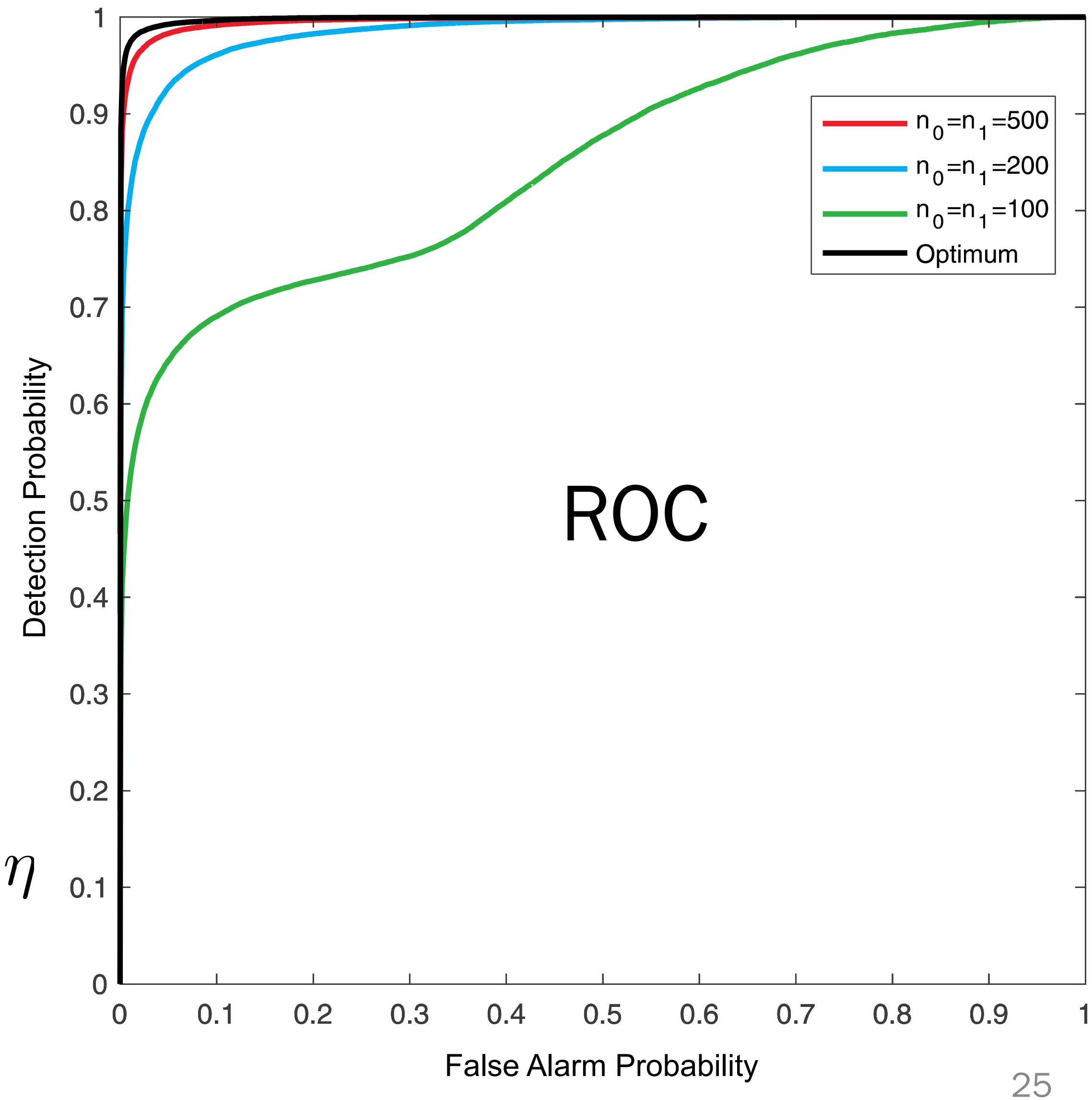
$\mathsf{u}_1(x_t, \theta_1) : \ 1 \times 10 \times 1$ (Exponential)

Training data $n_0 = n_1 = 100, 200, 500$

Testing $n = 20$, i.e. $\{x_1, \ldots, x_{20}\}$

$$\sum_{t=2}^{20} \mathsf{u}_2(x_t, x_{t-1}, \theta_2) - \sum_{t=2}^{19} \mathsf{u}_1(x_t, \theta_1) \underset{\mathsf{H}_0}{\overset{\mathsf{H}_1}{\gtreqless}} \eta$$

100000×20 samples from H_0 and H_1



What is Missing from the Analysis ?

- For fixed $\omega(\mathsf{r})$ find $\rho(z)$ that produces best estimates for $\omega(\mathsf{r}(X))$

  Requires analysis of estimation performance of neural networks for finite models (extremely challenging !!!).

- Rank functions $\omega(\mathsf{r})$ according to their approximation accuracy

  Is it possible to show that Mean Square is worse that Exponential or Cross Entropy?

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